UNIVERSITY OF ALBERTA

CONTINUOUS-TIME STOCHASTIC PROCESS CHARACTERIZATION AND VALUATION OF MINERAL INVESTMENTS AND SOFTWARE INTERFACE

By

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ABSTRACT

The economic evaluation of mineral projects forms the basis for project acquisition, investment and financing in the mineral sector. High capital intensity and irreversibility in the industry require the use of valuation methods that are rigorous and respond to market, industry and project risks. Conventional methods fail to properly account for these risks, their resolution through time and their effects on project value.

This study uses real options valuation method to develop two valuation models that consider both price and reserve uncertainty as opposed to models that treat reserve as known and constant. It also develops an interactive module (based on the developed models) that could be used as a tool by industry practitioners. This will allow option pricing theory to be used in valuing mineral projects without the need for thorough theoretical modeling.

The first model (2DPM) assumes that uncertainty in economic reserve is directly a result of flexible operational cut-off grade in response to price swings. The second model (2DPR) uses a more general model of reserves by assuming that the net relative change in reserve between evaluations follows a geometric Brownian motion. Results show that in general reserve uncertainty decreases the value of a mine compared with results from constant reserve model (CRM). Results from 2DPM show that the determination of cut-off grade independent of optimal operating policies may not always add value and delays the decision to invest. At a copper price of $1.5/lb, 2DPM (CRM) values a 127.8mt mine at $996

million ($1134 million) and suggests that, a critical price of $1.46/lb ($1.29/lb) of copper is high enough to justify the capital investment of $150million. The results from 2DPR show that although generally the value of the mine decreases ($952million), the decision to invest is made early (threshold price of $1.07/lb) when reserve uncertainty exists. This is attributed to the fact that the resolution of reserve uncertainty is achieved through investment and exploitation of the resource whereas under the assumption of constant reserve, uncertainty in price is resolved just by waiting.

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NOMENCLATURE

This section list all the symbols and abbreviations used in the text. They have been arranged in alphabetical order and not necessarily in the order in which they appear in the text.

1-D	One-dimensional
2-D	Two-dimensional
2DPM	Two-dimensional-price
2DPR	Two-dimensional-price reserve model
∞	Infinity
A	Periodic cash-flow from mine
Ac	Cost of maintaining a temporary closed mine
BCR	Benefit cost ratio
BS	Black and Scholes
C()	Value of a temporary closed mine
CAPM	Capital asset pricing model
CRM	Constant reserve model
C_S	First derivative of the call price with respect to price
C_{SS}	Second derivative of the call price with respect to price
C_t	First derivative of the call price with respect to time
CTSP	Continuous time stochastic process
D()	Value of a mine during development
DCF	Discounted cash-flow
DMV	Derivative mine valuation
DTA	Decision tree analysis
dw	Increment of the standard Wiener process governing grade
dz	Increment of the standard Wiener process governing price
F(S, t)	The price of a futures contract with maturity t, given a price S
FEM	Finite element method
F_S	First derivative of the futures price with respect to price

F_{SS}	Second derivative of the futures price with respect to price
F_t	First derivative of the futures price with respect to time
GBM	Geometric Brownian motion
Io	Initial investment capital
IRR	Internal rate of return
Kc	Cost of bringing mine to temporary closure
Kco	Cost of re-opening a temporarily closed mine
M()	Market value of a mine
MARR	Minimum acceptable rate of return
Max	Maximum of
M_R	First derivative of the mine value with respect to reserve
M_{RR}	Second derivative of the mine value with respect to reserve
M_S	First derivative of the mine value with respect to price
M_{SR}	2nd derivative of the mine value with respect to price and reserve
M_{SS}	Second derivative of the mine value with respect to price
M_t	First derivative of the mine value with respect to time
$\mathbf{n}$	the outward unit normal vector on the boundary.
NPV	Net present value
O()	Value of an operating mine
PDEs	Partial differential equations
PI	Profitability index
q	Production rate
R	Economic Reserve
r	The risk-free rate
ROA	Real option analysis
ROI	Return on investment
R_S	Sensitivity of reserve with respect to price
R_{SS}	Second derivative of the reserve with respect to price
S	Commodity or stock price
S_0^*	Critical price below which mine is abandoned in 2DPM
S_1^*	Critical price below which opened mine is closed in 2DPM

S_2^*	Critical price above which a closed mine is opened in 2DPM
S_a^*	Critical price below which mine is abandoned in 2DPR
S_c	Critical price where reserve is insensitive to price changes
S_c^*	Critical price below which opened mine is closed in 2DPR
S_i^*	Critical price above which investment is made in 2DPR
S_o^*	Critical price above which a closed mine is opened in 2DPR
S_u^*	Critical price above which investment is made in 2DPM
SRCF	Surplus riskless cash-flow
T	Terminal period for a contract or project
t	Time
$V()$	Value of a mine before investment
α	Expected rate growth or drift rate in commodity price
α_M	Equilibrium return on the market portfolio
α_P	Equilibrium return on project under CAPM
δ	Convenience yield
Δ	Number of futures contract to create a hedged position
ϕ	Growth rate in economic reserve
γ	Arrival rate in a Poisson process
η	Expected drift rate in economic reserve
λ	The market price of per unit of risk
λ_R	The market price of reserve risk
σ	Proportional standard deviation of changes in commodity price
σ_M	Standard deviation of market returns
σ_{PM}	Covariance of project returns with market return
σ_{RE}	Effective volatility of reserve
σ_{RG}	Volatility of reserve due to grade changes
σ_{RS}	Volatility of reserve due to price changes
Ω	Computational domain - the union of all subdomains –
$\partial\Omega$	Domain boundary
$\overline{\sigma}$	Volatility of change in reserve

1 INTRODUCTION

1.1 Background

Successful mining venture acquisition, operation and management require evaluation methods that respond to global market dynamics and provide investors and managers with relevant information to make strategic investment and operating decisions. The valuation of mineral projects forms the basis for any project acquisition, investment and financing decisions by both investors and financial institutions in the mineral sector. In the area of mineral policy planning and taxation, governments rely on some form of mine value as a basis for decisions. It is therefore important that methodologies employed in the valuation process be rigorous and capture as far as practicable all the intricacies associated with the venture. Investments in most mining ventures are huge and characterized by capital irreversibility, which compounds the need for thorough assessment and evaluation of opportunities prior to the commitment of funds.

Economic Darwinism characterizes today's business and market-driven economies in the sense that the ability of a firm to survive hinges on the choices it makes regarding the available investment opportunities. With limited financial resources, every firm is thus faced with the capital budgeting decision regarding which investment projects to commit funds. There are several investment analysis tools and decision criteria used by firms to evaluate and rank project profitability and viability. The most common are the payback period analysis and the discounted cash-flow (DCF) methodology. DCF is used in variant forms as net present value (NPV), benefit cost ratio or profitability index (PI) and internal rate of return (IRR) or return on investment (ROI). The methods used in capital budgeting have over the years become sophisticated. Whilst in the early 70s, firms using discounted cash-flow methods were characterized as using sophisticated methods (Schall, Sundem and Geijsbeek, 1978) these methods lack the capacity to deal with the complexities that exists in today's business environment. Over the years, methods such as the capital asset pricing model (CAPM)[1], decision tree analysis (DTA), simulation and most recently real option analysis (ROA) or derivative mine valuation (DMV) have evolved.

1.2 Statement of Problem

Mineral ventures have the unique characteristic of being location specific, with long-lead times, reserve uncertainty (the backbone of a resource firm) and a high volatility in output prices. These characteristics require that rigorous methodologies with comprehensive analysis be used to examine the decision to invest, the risk exposure and the return on investors' capital.

Conventional capital budgeting methods such as discounted cash-flow, one-period capital asset pricing (and other variations) are good investment decision tools when there is perfect information about the project and the relevant information is known with certainty. They are also good tools when the decision is completely reversible without any losses. In these situations there are no risks involved in the investment decision so that the opportunity can be valued by discounting future cash-flows at the risk-free rate. This is inconsistent with what is observed in today's business environment. Conventional methods therefore lack the built-in capacities to handle the strategic options and managerial flexibility embedded in projects in the presence of project uncertainties. They also fail to effectively account for the dynamics that underlie the value drivers in the valuation model. Conventional methods therefore take a static view of the investment decision problem, and fail to properly recognize the value added by management's ability to adapt to changes in market and site specific conditions to capture strategic value.

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[1] Market equilibrium model used to determine the appropriate relationship between risk and return.

Paddock *et al.* (1988) and McCormack *et al.* (2001) ascertained that the value accruing to owners of reserves upon exploitation is usually higher than that predicted from discounted cash-flow analysis, especially if there is undeveloped reserves.

Also for the same venture, conventional methods return different values depending on the risk characterization and preferences of investors, which (the risk preference) is reflected in the selected discount rate. The choice of the appropriate discount rate depends on managerial preferences and subjectivity and is not based on any quantitative model especially if the project risk is not priced by the market.

Derivative asset valuation or real options deals with these shortcomings. This approach is able to incorporate into the valuation model both the options available as well as the managerial flexibility at the disposal of management prior to and after the investment decision. The investment project is thus modeled as an option whose value is derived from the payoff of the options embedded in the project. The approach is based on the theory of financial option pricing which hinges on the ability to create a portfolio that is instantaneously devoid of risk. This riskless portfolio is created by replicating the option on an asset with riskless borrowing to finance the purchase of the underlying asset on which the option is priced (Black and Scholes (1973); Merton (1973); Cox and Ross (1976); Cox, Ross and Rubinstein (1979)).

The mineral resource valuation models developed thus far capture the stochastic behaviour of commodity prices, convenience yield and risk-free rate but the economic reserve base, which is also a major value driver for a mineral venture, is modeled as known with certainty. These models therefore take a "warehouse" view of mineral reserves. This assumption definitely limits the scope of these models given that cut-off grade revision and hence mineral reserve revision is a norm rather than the exception in the mineral sector.

Whilst the inadequacies of the traditional capital budgeting techniques are generally recognized by the mineral resource industry, the shift to the use of derivative asset valuation approach has been slow due to the 'complexities' of pricing the resulting physical option opportunities. This means that firms continue to make suboptimal investment decisions especially if the investment project has sequential interdependence with other embedded opportunities. There is therefore the need to develop a hands-on and user-friendly environment that allows industry practitioners to evaluate the investment opportunities as options without having to go through the theoretical underpinnings of option pricing.

1.3 Objective of Research

In the mining context, mineral reserve is the quantity of resource that can be extracted at a profit given an extraction cost profile and commodity prices. Ore reserve within a given property can thus change either because the marginal cost profile shifts over time or the price per unit changes or both. The reserve could also change due to the resolution of uncertainty surrounding the grade distribution within the ore body. The reserve base of a mineral property is therefore a random variable whose dynamics are determined by certain micro- and macro-state variables. The primary objective of this research is to develop a mineral resource valuation model that captures changes in mine value due to uncertainties in both commodity prices and the mineral reserves. This requires characterization of the stochastic processes that govern the evolution of both prices and reserves throughout the project. These stochastic processes form the basis for the development of equilibrium equations that describe the value of the mine. Since it is intended to find out how incorporating reserve uncertainty affects mine value, the developed model will be compared with the model that treats reserve as known and constant.

Another objective of this research is to offer industry practitioners the opportunity to seamlessly use derivative mine valuation methods for assessing asset value.

This research will therefore develop a user-friendly interface that uses the developed valuation models to determine the value of a mineral resource project. The successful development will help industry overcome one of the fundamental problems associated with the use of option pricing techniques in the evaluation of investment projects.

1.4 Scope and Limitations of Research

This research is limited to the use of the concept of portfolio replication with no arbitrage arguments[2] to develop a general partial differential equation that governs the dynamics of a mineral property at different stages of development. It can be seen as an extension of the now seminal work of Brennan and Schwartz (1985) and Frimpong (1992) in valuation of natural resource investments. This study develops a 'two-dimensional-price (2DP) and two dimensional price-reserve (2DPR) models to characterize the dynamic behaviour of the value of the mine in the price and reserve state space, as well as, in time space. The first model is described as 2DP model because the variation of reserve is attributed directly to change in economic cut-off grade as a result of only changes in prices; reserve is thus explicitly modeled as a function of price. In the 2DPR model, the uncertainty in the reserve could be as a result of uncertainty in grade, price or both. The results will be compared with the 1-D model that assumes no uncertainty in the reserve base.

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[2] Black and Scholes (1973); Cox and Ross (1976)

This study will apply stochastic control, portfolio hedging and replication theories in a continuous-time framework to develop a market and industry-responsive methodology to solve the problems associated with conventional methods. The limitation of this study is that the marginal cost curve, volatility of futures and commodity prices, as well as, the risk-free interest rate are assumed to be known and constant over the life of the mine. See Appendix *A* for details on *REALOPT*, the developed software.

A user-friendly interface is developed to allow the model to be applied to the evaluation of any mineral resource project that fits with the underlying model assumptions. A limitation of the developed user-interface is that the user is required to have both MATLAB and FEMLAB installed.

1.5 Research Methodology

The research has as its core an extensive literature review focusing on the different theoretical and practical project evaluation schemes and advances in option pricing theory, especially in the pricing of contingent claims. Mathematical models that govern the dynamics of a mineral venture, consistent with mining and finance practice, as well as economic equilibrium will be developed. Numerical models will be developed from the ensuing mathematical models to facilitate their solution using both MATLAB and FEMLAB as the platform for computer coding. The models will be used to solve for the value of a copper mine and the results compared with that from both the 1-D constant reserve model and the DCF technique. The model solutions are validated by comparing the solution to that from the net present value method. MATLAB will be used to develop the graphical user-interface.

1.6 Scientific and Industrial Contribution of Research

This research is geared towards developing a model that takes into account practical and realistic issues with mineral venture valuation. The reserve base of a mine changes as cut-off grade changes in response to price changes and yet current models do not account for this. The ability to account for variability in reserve makes the model more adaptable to the characteristics of mineral assets as opposed to constant reserve continuous-time models. Conventional methods are to a large extent inefficient; however they are still used largely because of their simplicity. The development of a user-friendly interface that allows the use of option pricing theory in valuing mineral projects (without the need for thorough theoretical modeling) will help industry overcome the basic problem associated with using real option techniques (see Appendix *A*). This will enhance competitiveness and a more efficient allocation of resources in the mineral resource sector.

In the academic and scientific community this work will push the research frontier in the use of real options for valuing mineral assets. As Laughton *et al.* (2000) put it; there has been less work in the area of incorporating unpriced risk (which includes local project-specific risks) in real options models.

1.7 Relevant Terminologies

The theory of option pricing and contingent claim analysis comes with a host of terminologies, thus it is worthwhile to review a number of relevant terms for completeness.

An *option* holder on an asset has the right but not the obligation to buy (*call option)* or the right to sell (*put option)* the asset at a predetermined price (*strike or exercise price)* at a given date (*maturity date* or *expiration date)*. The act of using this right is referred to as *exercising* the option. An *American option* can be exercised at any time up to the maturity date but a *European option* can only be exercised at maturity. American options thus offer more flexibility to the option holder and as a result are often more valuable. The person who agrees to buy the asset holds the *long* position and the counterparty holds the *short* position. *Arbitrage* is a risk-free transaction that involves an attempt to profit by exploiting price differences of identical or similar financial instruments, on different markets. A *derivative security* (*contingent claim)* is a financial instrument (like options and futures) whose value depends on the characteristics and value of certain basic underlying assets (like stocks, commodities).

1.8 Structure of Thesis

Chapter 2 of this thesis provides a detailed survey of the literature in the general area of project valuation with particular attention to the use of real options. Different traditional project valuation methods are examined, and their strengths and short-comings identified. The theory of option pricing is reviewed looking at the initial break-through by Black and Scholes (1973) and Merton (1973) as well as the different extensions and modifications. The paradigm shift from the traditional capital budgeting techniques to real option approach is examined with special emphasis on its use in the mineral resource industry. There is also a review of the different methodologies for solving the resulting equilibrium equations that characterize the value of a project. Chapter 3 reviews the fundamentals of commodity futures model and develops the general mine model; the models in Chapter 3 are used as the basis for developing the 2-D continuous-time price model and the 2-D continuous-time price-reserve model in Chapters 4 and 5. Chapter 6 concludes the thesis with key findings and recommendations and Appendix *A* discusses the development of the graphical user-interface.

2 LITERATURE REVIEW

2.1 Introduction

Capital budgeting is the process of identifying what long-lived assets a firm should invest in. Capital budgeting decisions determine the future growth and productivity of the firm. This process can either be simple or very complicated depending on either the characteristics of the project, the firm, the industry or the general macro-economic environment or a combination of these factors. The unique characteristics of the mineral industry put mineral venture evaluation into the complex category. This is due to the fact that mines are location specific, have long lead-times, require high capital injection and most of all resource endowment is non-renewable. The uncertainty associated with the inventory of resource, as well as, possible large volatility in mineral prices even adds to the complexity. The following reviews the general techniques used in project evaluation, the underlying theories and their ability or inability to capture the intricacies that may exist in a mineral project in particular.

2.2 Conventional Mine Investment Evaluation Methods

The discounted cash-flow (DCF) method forms the basis of the traditional investment evaluation and capital budgeting techniques. The DCF concept is based on the fact that future profits and expense are worth less than the same income and expenditure in the present because cash available now can be invested to generate additional income (Dulman, 1989). The approach requires the evaluator to estimate the expected yearly project net cash-flows and then to discount these cash-flows to account for timing and risk. The estimated net cash-flows (after any applicable taxes and deductions) are discounted by an appropriate discount rate. The sum of the discounted values constitutes the value of the asset. The requirement for this approach to be of practical use is that, the project or asset should generate some income over the life of the asset. The rationale is that an astute investor will not pay more for an asset than the income the asset can generate (Gentry and O'Neil 1984).

In the standard net-present-value (NPV) criterion, the net cash-flows are discounted by firm-wide minimum accepted rate of return (MARR) and then the project is accepted if NPV is positive. The *benefit cost ratio* (BCR) or *profitability index* (PI) is a variant form of the NPV and it is the ratio of the present value of the cash inflows to the present value of the cash out-flows. The decision rule is to accept projects with PI greater than one. An alternative criterion is the internal rate of return (IRR), which is the required discount rate that sets the NPV to zero. The project acceptance criterion is to accept all projects that have an IRR greater than a company-wide preset *hurdle* rate. Using the NPV or IRR to evaluate the acceptance or rejection of a project gives the same accept or reject decision. The IRR, however, has some problems when it has to be used as the decision criterion in selecting between competing projects. These problems relate to the fact that the projects under consideration may be of entirely different scale and also the timing of cash-flows from the different projects may be significantly different (Ross *et al.* (1999); Torries, (1998)). In such instances, it is recommended that IRR analysis be carried out on the incremental cash-flow between the competing projects.

In order to account for risk in a project, using any of these criteria, the discount rate is raised (arbitrarily) to reflect the risk perceived to be inherent in the project. To quantitatively measure risk, Sharpe (1964) and Lintner (1965) proposed the capital asset pricing model (CAPM). The CAPM builds on the model of portfolio choice developed by Markowitz (1959) known as the mean-variance-model. Finance theory advocates that the only portion of risk that is relevant for consideration in capital budgeting is the systematic risk, which is the residual risk after achieving full diversification (Ross *et al.*, 1999) and it is measured by the project *beta*[3]. Under the CAPM, risky assets should have an expected return equal to the risk-free rate plus a risk premium that is proportional to *beta* and consistent with the financial market equilibrium. Thus, risks such as technical and geological uncertainty of a project that have no correlation with the performance of the market have no market price.

Two main procedures for calculating the NPV of investment projects using the risk-adjusted discount rate emerge. Under the first, the certainty equivalent of all future cash-flows is discounted at the risk-free rate of return. In most situations this procedure is not easy to apply since it is difficult to estimate the certainty equivalent cash-flow. Cortazar and Schwartz, (1998) and McDonald and Siegel

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[3] Beta measures the relative volatility of an asset to that of the overall market. It is defined as the ratio of the covariance between the asset returns and the market returns to the variance of the returns on the market.

(1985)[4] suggest that when the uncertainty is due to only asset prices and there exists an active futures market for the underlying asset, then the futures price at maturity T is the certainty equivalent of the spot to be received at time T. The prices of futures can thus be used as proxies to the certainty equivalent cash-flows in the future. In the second approach to evaluate NPV of an investment project, the expected cash-flows are discounted at the risk-adjusted rate. The second procedure is the widely accepted and practical approach. However, Ang and Lewellen (1982) and Ben-Horim and Sivakumar (1988) noted that, by applying the capital market equilibrium risk-adjusted rates to future cash-flows from projects in the imperfect market of real assets, a bias is introduced in the calculated NPV. In particular, Ben-Horim and Sivakumar (1988) showed that the NPV is under-estimated when it is calculated in this way. They showed that the shortfall in value is due to the fact that surplus riskless cash-flow (SRCF) in excess of the investment capital is discounted at a higher discount rate instead of at the risk-free rate. They proposed a procedure (within the framework of CAPM) that removes this bias.

Myers and Turnbull (1977) also showed that if a project's future cash-flow is made up of more than one component (which could be looked at from the point of view of certain and uncertain components), then the project *beta* decreases over the life of the asset or project. This is consistent with mineral projects where the risks (especially with reserve) decrease over the life of the project. In this sense, using a single *beta* (risk-adjusted discount rate) for a project with multiple period cash-flows (typical of mineral projects) becomes inadequate. Fama (1977) and Constantinides (1980) introduced the multi-period CAPM to account for situations where the future project cash-flows are uncertain. Fama (1977) showed that the current market value of any future net cash-flow is the current expected value of the cash-flow discounted at risk-adjusted discount rates for each of the periods until the cash-flow is realized. The discount rates are known and non-stochastic, but the rates for the different periods preceding the realization of a cash-flow need not be the same, and the rates relevant for a given period can differ across cash-flows.

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[4] Although this is not explicitly stated in McDonald and Siegel, the observation that futures prices permit a firm to value a project without having to forecast future commodity price, even if the future price is a bias predictor of the future commodity price; and that using future commodity prices eliminates the need to adjust for systematic risk is in the same spirit as Cortazar and Schwartz.

Frimpong and Whiting (1998) developed a dynamic risk model for valuing long-term multi-phase projects. They recognized that for multi-phase projects especially mining ventures, the risk profile of the project declines from a maximum at the exploration phase up to a residual level during the production phase. They assumed that within each project phase, the change in the source and magnitude of risk is marginal so that the risk structure can be considered constant. They modelled the risk structure using a constant *beta* within each phase to generate the appropriate periodic discount rate. They used their model to value a gold venture and concluded that using a single discount rate under-estimates the project value by up to 63%.

Constantinides (1978) developed a general partial differential equation (PDE) (within the framework of CAPM market equilibrium) that must be satisfied by any project that generates uncertain cash-flows. One of the conditions imposed by this model is that there be a perfect market for the project, but not necessarily traded in the capital market. This condition will become useful later in the development of the 2-D equilibrium model underlying a mineral venture in the sense that reserve[5], which is a major value driver in mineral venture valuation, is not traded on the capital market.

2.2.1 Dealing with Risk and Uncertainty in Conventional Methods

Other methods that try to account for uncertainty in asset valuation are sensitivity analysis, simulation, risk-adjusted input parameters and decision tree analysis (Gentry and O'Neil (1984); Torries (1998)). Decision tree analysis (DTA) also attempts to account for the possibility of later decisions that could be taken by the firm's management. Trigeorgis (1996) argues that although these methods stop short of offering a manageable consistent solution, they help management improve its understanding of the investment decision.

The cash-flow estimates used in capital budgeting to determine NPV are derived from forecast of other primary variables (project life, production volume, costs and product price). Sensitivity analysis examines the impact of marginal changes in these forecasts on the NPV, one key variable at a time, with all others kept constant. The output from sensitivity analysis is usually the basis for further research into the individual input parameters, since one can quantify the effect of an error or uncertainty in an estimate. Sensitivity analysis has the limitation of treating the input variables as independent when in fact there are often interdependencies. A variant form of sensitivity analysis is scenario analysis which attempts to resolve the issue of interdependencies. Here the analyst constructs a set of different likely events (such as best case or worst case) and investigate their effect on the project value.

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[5] It can however be argued that if capital markets are indeed efficient then there is a market response (through the firm's share price) to announcements regarding reserve additions and discoveries and in that sense reserve is indirectly traded through the firm's stocks. Although this argument is strong for a purely exploratory company it is unrealistic to assume that reserve risk from a particular mine site can be hedged by trading securities.

The risk-adjusted input parameters approach tends to look at the technical uncertainty regarding the project. When considering the extensive uncertainty encountered in new mineral projects, industry executives often use conservative values for parameters such as price, mining costs, ore grade, output rate, recovery rate to account for risk (Gentry and O'Neil, 1984). Here there is the danger that compounded conservatism could render marginal projects infeasible.

Bey (1981) suggested the use of simulation when there is uncertainty associated with the project. With simulation or stochastic risk analysis, the critical input parameters (usually identified after sensitivity analysis) are assumed to follow a given distribution instead of using deterministic input parameters in the estimation of cash-flows. The probability distribution could either be discrete or continuous. The use of simulation to characterize uncertainty involves the repeated random sampling from the assigned probability distribution of each of the crucial variables (taking into account any correlation between the input variables) that affect the project value; in this way the project value becomes dependent on uncertain cash-flow variables (Hertz, 1964). The different project values corresponding to the input distributions can then be described by a probability distribution to characterize the risk of the project. In most cases the input probability distributions cannot quantitatively be determined and rely on analysts' subjective estimates, which is often a source of bias in the simulation results. Another possible source of bias is the intricate interdependencies that may exist between the input parameters and which may be difficult to capture with the simulation model.

Myers (1976) argued that *"If NPV is calculated using an appropriate risk-adjusted discount rate, any further adjustment will be double-counting. If a risk-free rate of interest is used instead, then one obtains a distribution of what the project's value would be tomorrow if all uncertainty is resolved between today and tomorrow. But since uncertainty is not resolved in this way, the meaning of distribution is unclear."*

The results from simulation also takes into account the idiosyncratic risks in the project that the CAPM 'fails' to capture. Lewellen and Long (1972) argued that the inability to reveal how the resulting NPV distribution interacts with the returns faced by the firm in other projects or by investors in their personal portfolio is the major short-coming of the simulation. This is however true only if the investment project is owned by well-diversified investors, who need only be compensated for the systematic component of the risk of the project.

Simulation is often described as *forward looking* (Boyle, 1977, Trigeorgis 1990) and so it may be inappropriate for solving path dependent problems. However, Cortazar and Schwartz (1998) and Longstaff and Schwartz (2001), in using simulation to value American type options, suggest that this is not a short-coming[6].

The decision tree analysis (DTA) is a suitable approach for analyzing sequential investment decisions when uncertainty is resolved at discrete points in time and management recognizes the interdependencies between the different aspects of the project. With this method, management is able to structure the decision problem by mapping out all feasible alternative actions contingent on the possible states of nature in a hierarchical manner. It also explicitly recognizes the interactions between the initial project investment decision and subsequent decisions in the future (Trigeorgis 1996). A decision tree analysis typically consists of four steps: (1) structuring the problem as a tree in which the end nodes of the branches are payoffs associated with a particular path (scenario) along the tree, (2) assigning probabilities to events represented on the tree, (3) assigning payoffs for consequences (dollar or utility value associated with a particular scenario), and (4) selecting course(s) of action based on analyses (using dynamic programming).

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[6] American type options are forward looking in the sense that a holder has to decide whether it is optimal to prematurely exercise the option or hold until maturity.

One draw back in this approach is that the tree can easily become unmanageable as the number of paths from a node increases. Also, real chance and decision events are often not resolved in discrete time but in continuous time. Whether to use risk-adjusted discount rate or risk-free rate is another limitation of the decision tree method. Notwithstanding these limitations, the ability of DTA to recognize that the value of an investment decision is in fact contingent on a whole array of other decisions is a plus. Trigeorgis and Mason (1987) clarified that option valuation (the novel valuation method) can be seen operationally as a special, economically corrected version of DTA.

Conventional methods are thus generally imprecise in valuing risky and flexible projects in the sense that a single risk-adjusted discount rate is usually used to discount the cash-flows from the project. By this approach, risk is valued at the level of the cash-flow instead of at the source. Also, an inherent assumption in these methods is that once a project begins it has to run throughout its course as originally set out without the possibility for any managerial and operating flexibility. Investment decisions are thus modelled as a now or never scenario.

To appropriately value the effect of uncertainty at the source requires that the different sources of risk be individually adjusted for their 'specific' risk and then discounted for time at the risk-free rate. Insights from the valuation of derivative assets (options) from financial markets allow us to create dynamic quantitative models that track the evolution of the underlying uncertainty through time.

2.3 Derivative Mine Valuation Method

Derivative mine valuation is based on asset pricing methods that use ideas originally introduced into financial markets by Black and Scholes (1973) and Merton (1973) in their analysis of stock options. Options on stocks and other financial assets are derivative instruments whose value depends on the value of certain basic assets like stocks or bonds. Complex assets can thus be valued as a dynamic combination of simple assets that replicates the complex asset. The theory of pricing these financial options is applicable to real assets in the sense that real assets also have their value contingent on some other basic underlying assets whose values are stochastic.

A variable whose state changes randomly with time is said to be stochastic. These changes in the state space could either be discrete or continuous and could also be discrete or continuous in time (Karlin and Taylor, 1975). There are two basic building blocks in modeling continuous time asset prices: the Weiner (diffusion) process and the Poisson process. The Weiner process is a real-valued continuous time stochastic process with small independent increments over time. The Poisson on the other hand is discrete in the state space but continuous in time with independent increment, but these increments are not over small time intervals. The Weiner process is often used if markets are perceived to be dominated by “ordinary” events and the Poisson process is used for modeling systematic jumps caused by rare or extreme events in the market (Neftci, 1996).

The stochastic behaviour of an asset (resulting from the stochastic characteristics of the underlying variables) is important in developing models for valuing the contingent claims on the asset. In most instances one of the above stochastic processes or a combination of both is used to characterize the uncertainty in the underlying state variables.

2.3.1 Theory of Option Pricing

The basic idea of pricing options on financial assets is that if markets are continuous and complete, then it is possible to 'create' derivative securities using dynamic portfolio strategies that are self financing[7]. By forming a portfolio consisting of the underlying asset(s) coupled with risk-free borrowing, and continuously rebalancing this portfolio, Black and Scholes (1973) and Merton (1973) showed that it is possible to replicate the payoff from a call option. Perfect replication however requires that markets are complete in order to find a portfolio and trading strategy that replicates the option in any state. Also if markets are complete and no-arbitrage conditions hold, then the option has a unique price irrespective of preferences. In the absence of arbitrage, the cost of setting up the portfolio consisting of the riskless asset and the underlying asset should be equal to the price of the option. This hinges on the principle that in markets with complete information, assets having the same payoff and risk structure must have the same price.

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[7] A self financing strategy has the property that the value of the replicating portfolio is equal to the value of the position being hedged without the need for infusion or withdrawal of cash. If this condition is satisfied at any time then the strategy is dynamic as well.

For a simple European call option on a stock, Black and Scholes (1973) assumed that the underlying stock returns follow a geometric Brownian motion (GBM), the underlying stock pays no dividends and volatility in returns is constant as illustrated in equation (2-1).

$$\frac{dS}{S} = \alpha \, dt + \sigma \, dz \qquad (2\text{-}1)$$

In equation (2-1), S is the stock price, α is the drift term or the expected rate of return on the stock, σ is the volatility of the stock and dz an increment of the standard Weiner process with mean zero and standard deviation $\sqrt{dt}$.

Further assuming that the risk-free rate is constant, there is no restriction on short selling, there is no transaction cost and taxes, the exercise price is constant over the life of the option and market operates continuously, they derived an equilibrium partial differential equation given by equation (2-2) that satisfies the price of the call option *C*. They found a closed formed solution in terms of the underlying stock price, time to expiration, volatility and the risk-free rate, r.

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0 \qquad (2\text{-}2)$$

Cox and Ross (1976) introduced the idea of risk-neutral valuation and provided an intuitive way to derive the option pricing formulas. They observed that since the Black and Scholes (1973) formula does not include α, the expected return of the stock, the formula is then valid irrespective of the average investor's risk preference. As a result, in a world satisfying all the economic assumptions of Black and Scholes (1973), such a world can be assumed to be risk-neutral and all assets should have an expected return equal to the risk-free rate. By assuming that the underlying stock price returns follow a lognormal distribution and replacing the drift term in the stock price stochastic process by the risk-free rate, they derived a closed form solution for a European call consistent with the Black and Scholes (1973) equation.

Harrison and Kreps (1979) showed that the absence of arbitrage implies the existence of risk-neutral probabilities associated with each possible payoff of the option at maturity. This probability measure is independent of the risk preferences and expectations of investors but depends only on the distributional assumptions of the underlying asset. Cox, Ross and Rubinstein (1979), used a discrete time model by assuming that the underlying stock price follows a binomial process. They developed the binomial pricing model using more economic intuition than the original Black-Scholes (1973) and Merton (1973) approach. They showed that the Black-Scholes (1973) model is a special limiting case of their simplified binomial model when the discrete time interval approaches zero.

There has been several modifications and extension of the Black-Scholes (1973) model. Merton (1976) and Cox and Ross (1976) relaxed the assumption of stock price continuity and derived option price for cases where the underlying stock price is characterized by a jump-diffusion process. Merton (1973) examined the case when interest rate is stochastic, exercise price changes and the underlying stock pays constant dividend even when there is the probability of exercise. Geske (1978) examined the effect when the dividend-yield is stochastic. Thorpe (1973) concluded that even if there is restriction on the full use of proceeds from the short sale, Black-Scholes (1973) formulation is still valid. Hull and White (1987) examined when the stock price volatility is stochastic and correlated with the stock price and found that there is a high degree that Black-Scholes (1973) over-prices options with long term maturities. All these extensions showed that there is no single assumption that is crucial to the Black and Scholes (1973) analysis.

Black and Scholes (1973) extended their general equilibrium model to value other contingent claims, specifically, equity of a levered firm. They observed that the payoff to equity holders on liquidation can be viewed as a call option on the value of the firm. Merton (1974) used the model to examine the risks on the value of corporate debt. Black (1976) used it to value commodity options, forward and future contracts. This option framework of valuation is useful for valuing real asset because the investment decision in real assets can be seen as contractual claims on the cash-flows (with option-like characteristics) generated by the assets.

2.3.2 Application of Options to Real Assets

The use of options in the valuation of real assets (real options) rose in response to the dissatisfaction of corporate practitioners and strategists with the inadequacy of the traditional methods Trigeorgis (1996). Traditional methods are generally unable to incorporate and capture value arising from strategic interaction of the different facets of a project as well as any managerial flexibility at the disposal of project managers.

In its general application to real assets, and in particular to investment in natural resource projects, the value of the project is contingent on the state of the project and the decisions made regarding whether the project should continue in its current state or not. The appraisal of a new project thus comes with several choices beyond simply accepting or rejecting the investment. Other choices such as delaying the decision until the market is favourable or deciding to start small and expanding later (if the result is positive) become available. There is value to draw from these additional choices but DCF analysis fails to account for these choices. As an illustration, the value of a mine will depend on whether it is in the exploration stage, development stage, and exploitation stage or temporarily closed.

In addition, the value at the exploration stage depends on whether it is optimal to develop that property or delay development. In this context, the value of a mine can be seen as a compound option; an option on a series of other options. By assuming a plausible stochastic process to characterize the uncertainty in the value drivers, it is possible to use option pricing theory to value all these nested set of options in every stage of the value chain. However with real assets, creating a perfect hedge by combining the underlying (or *twin asset)* and other primitive assets might not always be possible since the underlying assets are rarely traded. In such situations Vollert (2003) noted that the option values cannot be interpreted as market values but rather as lower limits as the value now depends on preferences.

Geske (1979 and 1977), Geske and Johnson (1984) and Selby and Hodges (1987) examined compound options on financial options, and they all provided close form analytical solution to the problems they examined in their studies. Longstaff (1990) looked at the pricing of options with extendible maturities where the option holder has the option to extend the exercise date and pay an additional premium (this value is established at the inception of the original contract) with an adjustment to the initial date's exercise price. He showed that compound options are special cases of options with extendable maturities. He cited its application to real estate options and a number of other financial instruments. This concept can also be applied in the valuation and granting of mineral exploration and lease rights where there is the possibility of extension of the exploratory period if the holder of the concession pays an additional fee to a governmental agency.

Application of real option to the investment decision is able to resolve the issue of optimal timing for expending the initial upfront investment capital for a project or the optimal time to discontinue with a project. Usually a firm will know the present value of future net cash-flow if it invests in a project today but this present value may be different if the project is deferred. The extra value or loss in value revolves around the uncertainties associated with some of the project variables. For instance the time to carry out exploration and development is not transparent as it is arbitrarily chosen when doing DCF analysis, as such there could be divergence in valuation between the two parties trying to reach a deal on the value of a property leading to bargaining failure. This aspect of the investment decision is analogous to determining the optimal strategy for the exercise of American type options on common stocks and other instruments. Baldwin (1982), Baldwin and Meyer (1979), Venezia and Brenner (1979) and McDonald and Siegel (1986), Brennan and Schwartz (1985), Ingersoll and Ross (1992), Frimpong (1992) and Cortazar and Schwartz (1997 and 1998) analyzed the investment timing problem under risk and uncertainty.

McDonald and Siegel (1986) showed that the net present value rule of "invest if NPV>0" is only optimal when the variance of the present value of future benefits and costs is zero or if the expected growth rate in benefits (the drift in the stochastic process governing benefits) is minus infinity. They concluded that there is value added in waiting on a project and that the project should be carried out if and only if the present value of benefits exceeds costs by a threshold.[8] Baldwin (1982) suggests that this "NPV premium" requirement is necessary (especially when capital is not readily reversible) in order to compensate the firm for the loss in the value of future opportunities implicit in accepting a particular investment today.

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[8] For reasonable parameter values they estimated that the ratio of value to cost should be two.

Ingersoll and Ross (1992) suggested that making an investment today precludes the opportunity to take on the same project in the future. In that sense the project competes with itself in the future and so the generalised NPV rule is inadequate. In particular, if the yield curve (the term structure of interest rate) shifts through time in an uncertain manner, then it might be optimal to wait on an investment even if there are no uncertainties with other project parameters.

McDonald and Sigel (1985) studied project evaluation when the firm has the opportunity to shut down production. They recognized that, the ownership in the physical capital employed in production in a firm is equivalent to holding a call option on the producing firm. As a result, the owner can shut down (not exercise the option to produce) and avoid losses if variable cost of production exceeds sales revenue. They developed a market equilibrium model similar to that by Constantinides (1978).

Abel *et al.* (1996) examined the role played by reversibility and expandability in a dynamic model of optimal investment under uncertainty. They examined the firm's decision to add to its capital stock by investing in assets with the possibility of dis-investing (put option), and the decision to later expand by acquiring additional capital (call option)[9]. They found that the call option reduces the firm's current incentive to invest although it adds value to the firm. Depending on the future capital cost structure this value is extinguished by future investments. The put option however increases the incentive to invest. Since this put option is almost non-existent in the mineral sector because of the high capital irreversibility, this may explain the general low incentive in investment in mineral projects and the demand for higher returns by investors.

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[9] Even if either re-sale value is less than the original acquisition cost or if the future cost of capital acquisition is more than current cost of capital.

Brennan and Schwartz (1985) extended the theory of option pricing in the evaluation of a mineral resource. Using a one-factor price model[10], and assuming that the reserve level is known, they developed the equilibrium condition and the partial differential equation that must be satisfied by the value of a mine using the concept of portfolio replication. They explicitly accounted for managerial control on output rates, which they assume to respond to output prices[11]. Their model also recognized some inherent flexibility like the possibility to temporarily close down a project or even abandon it if output prices fall far below a certain threshold during the production phase.

Paddock *et al.* (1988) used the option approach within the framework of Brennan and Schwartz (1985), to determine the value of petroleum lease contracts. They emphasized that, embedded in any approach to valuing petroleum lease, there should be a rule that specifies when and if a firm should explore and develop a leased property (i.e. to exercise its options). This is usually difficult to establish especially using conventional DCF techniques. Their model also studied the effect of exploration and development costs and lags, and relinquishing[12] requirements on exploration and development investment-timing decision. They also looked at the lease as an option on exploration and development. They did not explicitly model the extraction option as they assumed this option is already embedded in the 'observed'[13] current market value for a developed tract.

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[10] Price model is described as one factor since the volatility, expected return and convenience yield of the commodity price stochastic processes are assumed constants.

[11] In practice however, output rates cannot be allowed to vary with output prices given the high capital intensive nature of mining projects. For high fixed cost firms, production levels required for break-even is often closer to capacity than for low fixed costs firms. This explains why mines sometimes employ 3 shifts a day and seven-day-per week schedules. See Gentry and O'Neil (1984).

Cortazar and Schwartz (1997) and (1998) developed similar models for the valuation of an undeveloped oil field assuming a one factor and two factor models respectively for the stochastic behaviour[14] of oil prices. The two factor model allows them to capture both the upward sloping (contango) and downward sloping (backwardation) term structure of futures prices. Although they explicitly model the extraction option they again assume the reserve level is known.

Frimpong (1992) examined the value of typical mineral venture (within the framework of Brennan and Schwarz) with metal price and reserve as the state variables by determining optimal operating and investment policies that maximize the value of the mine. He also developed a 2-D model (in price and expected ore grade uncertainties) to examine the use of feasibility study in mineral venture development. He studied optimal feasibility study and its impact on ore grade expectation and uncertainty, required investment and overall mine value.

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[12] Relinquishing requirement is here analogous to the time to expiration in an option pricing model since lease contracts on mineral properties often stipulate how long a firm can hold the property before exploring and developing it.

[13] They suggest that a secondary market exists for developed reserves so it is possible to abstract what the market is willing to pay for a given developed reserve with a certain reserve level, extraction rate and cost structure. This brings up the same issue which comes up with the use of CAPM; whether any two projects are the same and whether the *beta* of one project can be abstracted from another project that has run its course in the market place.

[14] They assume a mean reversion process for both the oil price and the convenience yield. In an earlier work by Gibson and Schwartz (1989, and 1991) and Schwartz (1997) they found strong empirical evidence in favour of the mean reverting pattern of convenience yield.

Frimpong, Laughton and Whiting (1991) considered feasibility study management and investment timing in a mineral project. They demonstrated how increasing the geological knowledge about a mineral property through a feasibility study option can be used in addition to investment timing and other operating options to maximize the value of a mineral project. One can however observe that if the feasibility period matches the waiting period then the investment timing option (wait option) is embedded in the feasibility study option. Feasibility study thus adds value only when the cost of carrying out the study is less than any perceived value to be added from the study.

Schwartz (1997) extended Brennan and Schwartz (1985) by using both a two-factor stochastic-convenience-yield and three-factor stochastic-interest-rate price models[15]. He found a closed form solution for the value of the futures contract and used it as basis for the development of the equilibrium resource model. He further showed that the two-factor model can in fact be "collapsed" into an equivalent one-factor model. Whilst this is computationally easier to solve in the sense that there is only one state variable, the resulting solution also retained most of the characteristics of the more complex two-factor stochastic-convenience-yield model. He concluded that in the evaluation of long-term resource projects (copper and oil) it is important to consider mean reversion in prices as this is a better predictor of the term structure of future prices.

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[15] In the two factor models they assume commodity prices to follow the Geometric Brownian motion and convenience yield follows a mean reversion process. For the three factor model they assume there is also mean reversion in interest rates.

Samis (2001) introduced a project structure model that reflects the heterogeneous nature of mineral deposits in which he subdivided the deposit into zones differentiated by size, quality and location. He characterized the project as a portfolio of real assets in which each mining zone constituted a fraction of the entire portfolio. The project was operated in discrete intervals by choosing (at the start of each interval) an operating mode from a set of competing operating modes. Each mode specified the combination of zones that will be active and the amount of project capacity that is built, abandoned or closed temporarily. This procedure captured the value-added to a mine as a result of the geological structure of the deposit. It also accounts for any additional value as a result of efficient production planning during production and staged construction during the development phases. Although this procedure accounts for the heterogeneous nature of mineral deposits, the reserve state is still modeled as known and constant.

Cortazar *et al.* (2001) attempted to develop a real option model to take into account technical (geological) and price uncertainty. The model collapsed a two factor model into a one factor model by defining a new value function as a product of the two underlying state variables. They examined technical uncertainty only during the exploration stage and assumed that, the only relevant source of uncertainty during development and operation of the mine is price. The mine is however valued as the expected value of alternative mines whose characteristics are conditioned on the outcome of exploration. Each possible mine is valued using the Brennan and Schwartz (1985) model. They found that a significant percent of total project value could be attributed to development and the exploration options available to project managers, especially if the expected deposit is small. Smith and McCardle (1998) also looked at uncertainty in oil production rate and price and combined the two factors into a one factor revenue model. They introduced an integrated valuation technique that combined the use of real option and decision tree analysis.

2.4 Solution Procedures

In most cases the resulting equilibrium partial differential equations of the option valuation models do not have analytical solutions. In almost all cases a numerical approximation to the solution has to be constructed subject to the appropriate boundary and initial conditions. The solution procedure becomes even more difficult when one has to determine an optimal exercise strategy as part of the solution. This is often the case when the option to be evaluated is path dependent and has embedded American option characteristics. In such situations, one has to determine the free boundary as part of the solution. Some of the solutions procedures proposed and used are Monte Carlo simulation, finite difference method, and the binomial trees.

The use of Monte Carlo simulation to solve for the value of an option was first used by Boyle (1977). It is based on the recognition by Cox and Ross (1976) that, as long as a hedged position can be constructed, the option value can be obtained by discounting the expected maturity value of the option at the risk-free rate. The Monte Carlo approach therefore simulates terminal values of the underlying state variable (using a given stochastic process)[16] which can then be used to calculate the option value. The expected value of the terminal option values is then discounted at the risk-free rate to get the value of the option. The potential short-coming relates to the large number of simulation trials required to stabilize the variance of the terminal option values. The control variate approach has proved to be effective in increasing the accuracy of solutions (Boyle, 1977). The technique is simple and flexible in the sense that it can easily be modified to account for different processes underlying the state variables as well as account for different complex payoff structures, Geske and Shastri (1985). In addition, quite a number of state variables can be considered at the same time with no constraint. Other methods like the finite difference method limits the state variables to a maximum of three for any practical modeling.

Finite difference methods for numerically solving partial differential equations involve replacing all derivatives by a corresponding difference approximation. The domains of the state variables are equally spaced to generate grid points. The solution process involves finding an approximate solution at every grid or nodal point that satisfies the difference equations. The difference approximation scheme used could either be central difference, forward difference or backward difference or a combination. It is important however to ensure that the resulting difference equation is consistent with the original differential equation.

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[16] In this case, the drift term in the governing stochastic process is replaced with the risk-free rate in order to be consistent with risk neutral valuation.

The finite difference approach for valuing options was pioneered by Schwartz (1977) and Brennan and Schwartz (1978) and modified by Courtadon (1982) and Hull and White (1990). The different finite difference schemes are explicit and implicit methods. The explicit scheme relates the value of a derivative security at time t to three known values at time $t + \Delta t$. Implementation by this scheme is simple but could suffer from severe numerical instability if the choice mesh size[17] is not appropriate. The implicit method on the other hand requires the solution of a system of equations by iterative methods at each time step. Here, a known value of a derivative security at time $t + \Delta t$ relates to three unknown values at time t. Although the implicit method is more difficult to implement than the explicit method, the resulting algorithm is numerically stable.

Geske and Shastri (1985) showed that there is numerical efficiency in the rate of convergence by using the log-transform of the underlying state variable. A number of implementation algorithms ensue depending on the difference scheme that is employed. These algorithms include the backward implicit formulation, alternating direction explicit and implicit (ADE and ADI) and Crank Nicholson algorithm. Brennan and Schwartz (1978) showed that approximating the Black-Scholes partial differential equation by the use of finite difference approximation is equivalent to approximating a diffusion process by a jump process. The jump process is the assumption underlying the binomial lattice or binomial tree solution procedure. As noted earlier, the shortcoming of the finite difference method is that the solution is not easily tractable once the state variables increase beyond two.

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[17] Discretization interval in time with respect to the interval in other state variables defines the size of a typical mesh.

The binomial lattice approach was suggested by Cox, Ross and Rubinstein (1979). The underlying stochastic process is approximated with a simple binomial process over a short time interval with the corresponding risk neutral probabilities for the up tick and down tick. A lattice structure is set up and a dynamic program algorithm used to evaluate the value of the options at all the nodes. The value of the derivative security is known at the end of the structure (time *T*) so by moving backwards through the lattice the value at time zero can be evaluated. Boyle (1986) extended this by assuming a trinomial model. Boyle (1988) again extended the binomial model to account for the case when the value of the derivative security is controlled by two state variables. Hull and White (1988) proposed the use of control variate technique to improve efficiency in the solution.

Geske and Shastri (1985) compared the different approximation methods for valuing contingent claims. They conclude that the binomial approximation may be more efficient for the evaluation of smaller option values; however finite difference methods may be more efficient when the option values under consideration are larger. These conclusions were however based on only one state variable.

2.5 Summary and Conclusion

A detailed survey of the literature in the area of project valuation in general and in particular the use of real options has been carried out. This included extensive examination of the different traditional project valuation methods, their strengths and their short-comings. Conventional methods, in general, lack the capability to account for flexibility and uncertainty. The theory of option pricing was also reviewed looking at the early works and the different extensions and modifications. There has been the need for a shift from traditional capital budgeting techniques to techniques that deal with their short-comings.

The real option approach seems to be the new paradigm for the way forward. In its application to the evaluation of natural resource investments, little has been done to incorporate project specific risks like geological risk or reserve uncertainty in the valuation models thus far. Thus the current models do not account for one of the basic characteristics of the resource industry. These project-specific risks or uncertainties can however constitute an important input to the mine investment decision.

The purpose of this thesis is to account for one of such uncertainties; ore reserve uncertainty in mineral ventures. One of the main obstacles in the use of option pricing theory in mineral valuation is the mathematical skills required. This study bridges this gap by developing a user-friendly interface, which allows the use of option pricing theory to value mineral projects without the need for these mathematical skills. This will help industry overcome the basic problem associated with using real option techniques and will enhance competitiveness and a much more efficient allocation of resources in the mineral resource sector. The Chapters that follows describes two models that account for reserve uncertainty in the real option framework.

3 Continuous-Time Stochastic Process (CTSP) Model

3.1 Introduction and Modeling Philosophy

The reserve base of a mine constitutes one of its major assets. The component of reserves in a periodic cash-flow modelling is the quantity of ore that can be mined within a period subject to strategic and tactical plans and at a profit. Thus, there is an economic dimension to the definition of reserve for project value and viability. As indicated in Figure 3-1, the uncertainties in the "modifying factors" affect the level of economic reserves.

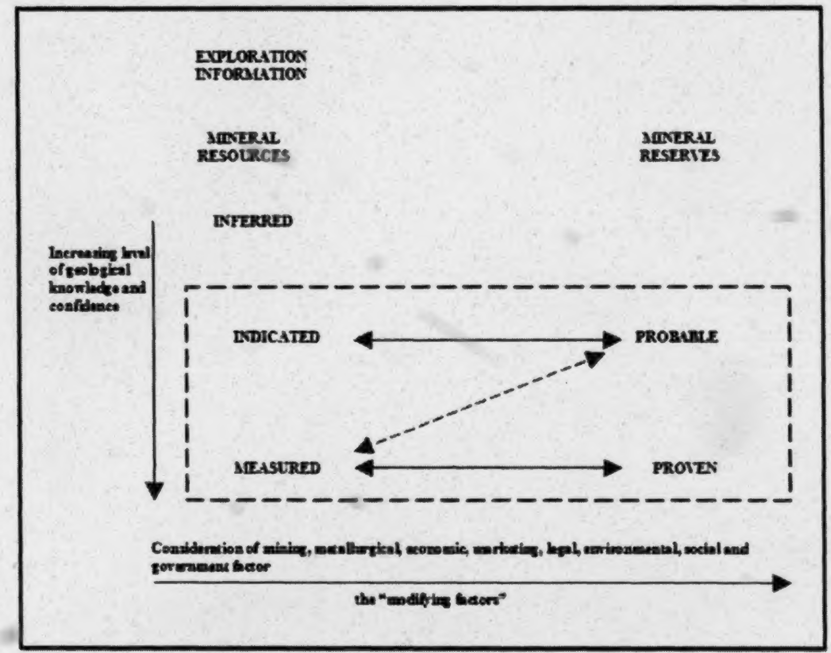
EXPLORATION INFORMATION
MINERAL RESOURCES
MINERAL RESERVES
INFERRED
Increasing level of geological knowledge and confidence
INDICATED
PROBABLE
MEASURED
PROVEN
Consideration of mining, metallurgical, economic, marketing, legal, environmental, social and government factor
the "modifying factors"

Figure 3-1: Mineral resource and reserve classification[18]

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[18] Source: CIM definition standards. http://www.cim.org/committees/StdsApprNov14.pdf (assessed December 12, 2005).

The viability of a project, in terms of investment for property development, is definitely a different problem from determining the value of a tract of mineral land. Ore body delineation and the block model generation are complete at the stage of investment decision on development. The economic reserves depend on the cut-off grade, which in turn depends on the mining and processing costs, as well as the commodity price. The uncertainty in reserve could either be as a result of estimation variance or human and/or systematic errors. Variation in commodity price could also affect ore and waste classification. Changes in classification could be augmented by the combined variation in price and grades. The sensitivity of reserve to costs and commodity price is mine-specific, and thus, exogenous to the mine valuation problem.

A variable whose state changes with time and/or space is said to be stochastic. The changes are said to be discrete in the state space if the variable can assume only a prescribed set of values (usually integer or categorical values). They are described as continuous in space (real-valued) if the variable can assume an infinite number of values. The changes in the state variable could also be either continuous or discrete in time (Karlin and Taylor, 1975).

The stochastic behaviour of an asset (resulting from the stochastic characteristics of the underlying variables) is important in developing models for valuing contingent claims on the asset. There are two basic building blocks in modeling continuous time asset prices: the Weiner (diffusion) process and the Poisson (jump) process. In most instances one of the above stochastic processes or a combination of both is used to characterize the uncertainty in the underlying state variables. The Weiner process shown in Figure 3-2 is a real-valued continuous-time stochastic process with small independent increments in value over time. The Poisson process (shown in Figure 3-3[19]) on the other hand is discrete in the state space but continuous in time with independent increments but these increments may not necessarily be over a small time interval.

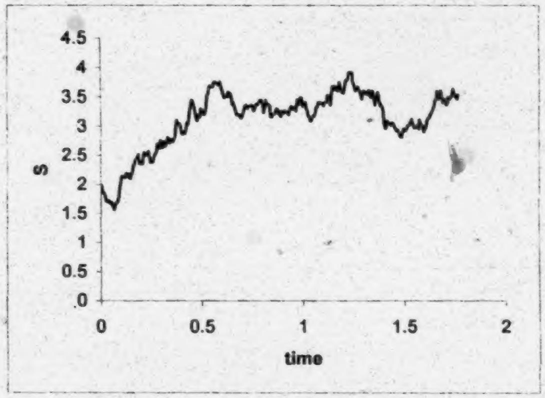

Figure 3-2: The Weiner Stochastic Process

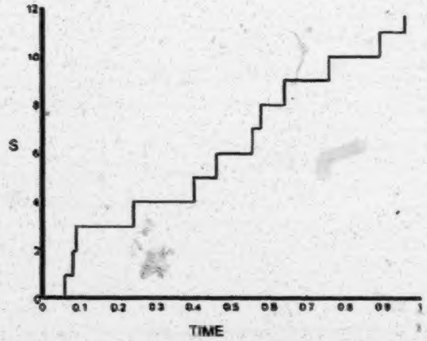

Figure 3-3: The Poisson stochastic process

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[19] Source: Neftci (1996); page 151.

Stochastic processes can be decomposed into two parts: a deterministic or drift term and a random component which are related through a stochastic differential equation. The deterministic part is the expected instantaneous change and the stochastic part causes volatility around the drift term. The Weiner stochastic process is a normally distributed variable with a mean of zero and variance proportional to the time lag. The standard deviation controls the extent of volatility in addition to the deterministic component (Chriss, 1997). The Weiner process is an appropriate model if the changes in the underlying variable are perceived to be dominated by 'normal' events so that in the short run there are no 'surprises' (Cox and Ross 1976). For commodity or stock prices 'normal' events may be a result of a temporary imbalance between supply and demand, changes in capitalization rates or changes in economic outlook (Merton, 1976).

The Weiner process stipulates that the current value of the variable fully reflects all past information and that its future value is influenced by present conditions alone and independent of past data. This is analogous to the semi-strong form of the efficient market hypothesis.[20] The models often used in modeling the Weiner process differ only by the nature of the drift and the diffusion coefficient. The coefficients could both be constants and independent of the state of the underlying variable (linear constant coefficient model) or both could be proportional to the state of the variable (geometric or square root model). The Weiner process is also described as mean reverting if the drift term is such that the long run value of the state variable reverts back to the drift term (Neftci, 1996). For consumption commodities, the mean reverting process is believed to be a better model for the price process (Gibson and Schwartz,1991; Schwartz 1997).

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[20] A market is said to be semi-strong efficient if investors cannot make an abnormal profit by investing in a stock upon the release of public information since this new information is immediately incorporated into the price of the stock.

The Poisson process is used for modeling systematic jumps caused by rare or extreme events in the market (Neftci, 1996). In a simple jump process the stochastic component is a Poisson distribution with an instantaneous probability, γdt, that there will be a jump[21] in the underlying variable. Although the probability of a jump approaches zero as the time interval approaches zero the size of the jump does not vanish. This is a major difference between the jump process and the Weiner process where, the variance of the change in the variable approaches zero as the time interval approaches zero (Neftci, 1996). Extreme events causing jumps (in discrete and random interval of time) could be due to the release of abnormal information specific to an industry, a company or a project; the onset of war in producing economies, business embargo and/or other producing governments' initiatives.

For this thesis it is assumed that the mine produces a single commodity whose price follows the stochastic process shown in equation (3-1).

$$\frac{dS}{S} = (\alpha - \delta)\, dt + \sigma\, dz \tag{3-1}$$

---

[21] The size of the jump could itself be stochastic.

$$dz = \varepsilon\sqrt{dt} \qquad (3\text{-}2)$$

In equation (3-1), S is the commodity price, α is the drift term or the expected rate of return on the commodity, σ is the annualized volatility of price and dz an increment of the standard Weiner process with mean zero and standard deviation $\sqrt{dt}$. δ is the convenience yield and is assumed to be constant.[22]. The increment of the standard Weiner process is given by equation (3-2); ε is a random sample from a standardized normal distribution.

3.2 Fundamental Theory of Commodity Futures Model

The futures model is important in derivative mine valuation because the futures market for a commodity is more liquid than the market for the commodity itself. As such it is practical to use a futures contract to hedge a position in the mine than the use of a contract on the commodity itself.

The futures price is the price (determined at the time of transaction) for the delivery of a unit of commodity at a defined future date. This price depends on the current price of the commodity and the time remaining to honour the contract. The futures contract, written on a commodity, is thus a derivative security whose value is contingent on the price of the underlying commodity. It is therefore possible to create a portfolio that consists of an appropriate position in a number of futures contracts and in the underlying commodity that will be instantaneously risk-free (Black and Scholes 1973, Merton 1973).

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[22] The convenience yield is the value that accrues to one holding a commodity instead of the futures contract. It can also be seen as the rate of return short fall from the equilibrium return that an asset with equivalent risk will earn.

If $F(S, \tau)$ is the futures price for the delivery of one unit of the commodity at time t for time remaining $\tau=(T-t)$ to maturity T, then by Ito's Lemma[23], the instantaneous change in the futures price is given by equation (3-3).

$$\mathrm{dF} = \frac{\partial F}{\partial t}\mathrm{dt} + \frac{\partial F}{\partial S}\mathrm{dS} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(\mathrm{dS})^2 \tag{3-3}$$

The notation shown in equation (3-4) will be used in this thesis.

$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= F_t \\ \frac{\partial F}{\partial S} &= F_S \\ \frac{\partial^2 F}{\partial S^2} &= F_{SS} \end{aligned}\right\} \tag{3-4}$$

It is important to note that within a small time interval, dt, higher powers of dt are zero but $(dS)^2$ does not vanish due to the properties of equation (3-2).

$$\left.\begin{aligned} \mathrm{dtdz} &= \mathrm{dt}^2 = 0 \\ \mathrm{dz}^2 &= \mathrm{dt} \end{aligned}\right\} \tag{3-5}$$

Substituting equation (3-1) in equation (3-3) and using the conditions in equation (3-5) and substituting $-F_\tau$ for F_t gives equation (3-6)[24].

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[23] See Section B2 of Appendix B for details on Ito's Lemma

$$dF = (-F_\tau + \frac{1}{2}F_{SS}\sigma^2S^2)\,dt + F_S dS \qquad (3\text{-}6)$$

The uncertainty in income (as a result of uncertainty in price) from the production and sale of one unit of commodity in the physical market can be hedged by taking Δ number of short positions in a futures market. The value of this portfolio, *P1*, is given by equation (3-7). The instantaneous change in the value of this portfolio including any convenience yield is given by equation (3-8). Equation (3-8) states that the change in the value of the portfolio is the algebraic sum of changes arising from the commodity price, the futures price and any additional benefit accruing from holding the commodity within that time interval.[25]

$$P1 = S - \Delta F \qquad (3\text{-}7)$$

$$dP1 = dS - \Delta dF + S\delta dt \qquad (3\text{-}8)$$

By substituting equation (3-6) for *dF* into equation (3-8) and simplifying gives equation (3-9).

$$dP1 = (1 - \Delta F_S)\,dS - \Delta(-F_\tau + \frac{1}{2}F_{SS}\sigma^2S^2)dt + S\delta dt \qquad (3\text{-}9)$$

By setting Δ to $1/F_S$ in equation (3-9), the random component of the change in the portfolio value disappears from equation (3-9) and the change in the value of the portfolio becomes deterministic. In a market with complete information with

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[24] Since $\tau = T-t$, $F_\tau = -F_t$

[25] The negative sign in equation (3-7) indicates a short position. The total convenience yield is assumed to be proportional to the commodity price.

no arbitrage opportunities, the return on this riskless portfolio should be equal to the risk-free rate. Since there is no upfront cash payment at the inception of a futures contract, the return on the portfolio is given by equation (3-10). Substituting equation (3-9) into equation (3-10) gives equation (3-11).

$$\frac{dP1}{S} = rdt \tag{3-10}$$

$$rdt = -\frac{1}{SF_S}\left(-F_\tau + \frac{1}{2}F_{SS}\sigma^2 S^2 - F_S S\delta\right)dt \tag{3-11}$$

The futures price therefore satisfies equation (3-12) subject to equations (3-13) to (3-15), (Brennan and Schwartz, 1985).

$$\frac{1}{2}F_{SS}\sigma^2 S^2 + F_S(r-\delta)S - F_\tau = 0 \tag{3-12}$$

Equation (3-13) states that, at maturity the futures price converges to the spot price. Equations (3-14) and (3-15) are the low and high metal price conditions respectively. Equation (3-14) states that the futures price collapses when the commodity price is zero and equation (3-15) states that the futures price is linear in price at very high prices.

$$F(S,T) = S \tag{3-13}$$

$$F(0,\tau) = 0 \tag{3-14}$$

$$F_{SS}(\infty,\tau) = 0 \tag{3-15}$$

Equation (3-16) is the analytical solution to equation (3-12); see Section B1 in the Appendix B for the analytical solution.

$$F(S,\tau) = Se^{(r-\delta)\tau} \tag{3-16}$$

3.3 Stochastic Mine Value Model

The market value of a mine, *M*, depends on the quantity of economic reserve *R*, the price of the underlying commodity *S* as well as the state of the mine as shown in equation (3-17).

$$M = M(S,R,t;j) \tag{3-17}$$

Where, *j* is a parameter that describes the state of the mine as shown in equation (3-18). The state of the mine could either be opened, closed, undeveloped or abandoned.

$$j = \begin{cases} o; \forall \text{ opened mine} \\ c; \forall \text{ closed mine} \\ u; \forall \text{ undeveloped mine} \\ d; \forall \text{ developing stage} \\ a; \forall \text{ abandoned mine} \end{cases} \tag{3-18}$$

At a given price and reserve, the value of a particular mine in the development stage will be different from another one at the production stage. This results from the difference in the extent to which uncertainty of the underlying state variables affect project values at different stages of a project. If *M(S, R, t; j)* is the market value of a mine, its instantaneous change due to changes in the underlying state variables can be formulated.

Using Ito's Lemma, the instantaneous change in the value of the mine is given by equation (3-19).

$$dM = M_R dR + M_S dS + \frac{1}{2}M_{RR}(dR)^2 + \frac{1}{2}M_{SS}(dS)^2 + M_{SR} dS dR + M_t \qquad (3\text{-}19)$$

It is clear from equation (3-19) that the change in the value of the mine is determined by the kind of processes that govern the evolution of the underlying price of the commodity as well as the reserve. The two Chapters that follow develop two different reserve processes and together with the price process assumed in equation (3-1) determine the equilibrium equation that satisfies the value of the mine in different states.

4 2D CTSP PRICE INVESTMENT MODEL – (2DPM)

4.1 Introduction and modeling Philosophy

This model is based on the assumption that the ultimate reserve base is known given a mill cut-off grade[26]. However, market conditions reduce this base to a level that can be mined and processed at a profit. The model is thus described as a 2D price model (2DPM). Management thus have the flexibility to affect the level of reserve merely by changing operating cut-off grade. An important input to this model is the tonnage-cut-off grade curve (which also has the tonnage-average grade curve) determined by ordinary kriging or any other geostatistical method used to estimate reserve. It is assumed that, for a given cost profile and recovery efficiency, management chooses operating cut-off grade in such a way as to break even from operations. In this sense, cut-off grade operating policy is exogenous to the model. For a given cost profile, one can therefore generate a corresponding tonnage-price curve and average grade- price curves from the given tonnage-grade curve.

If the marginal production cost curve is assumed to be known and constant, then any change in reserve will be as a result of change in commodity price which can be determined from the reserve-price sensitivity curves[27]. If the mine is in production, then in addition to this change from price fluctuations there will be changes due to periodic output. By defining a process that describes the state of reserve at any time the value of the mine can be determined.

---

[26] Reserve corresponding to a technical mill cut-off grade assuming no extraction and mining cost is used instead of that corresponding to zero-cut off grade since the mill puts a further constraint on what quality of material can be processed. The reserve, when mineral price far exceeds the processing and mining costs will approach this level of reserve.

[27] The term "reserve-price sensitivity curves" is used in this thesis to refer to both the first and second derivatives of reserve with respect to price while "reserve-price sensitivity curve" refers to only the first derivative.

4.2 Reserve Process

If production rates are known then the uncertainty in the change in "economic" reserve (which will result from flexibility in choice of cut-off grade) is governed by the same stochastic process that characterizes the price process. R in equation (4-1) is the reserve, q is the production rate and S is the commodity price. Using Ito's Lemma, the instantaneous change in reserve is given by equation (4-2). In equation (4-2), R_s is the slope of the reserve-price curve described here as the marginal sensitivity of reserve to price and R_{ss} is the slope of the reserve-price sensitivity curve and also described here as the shape factor. The sign of R_{ss} determines whether the reserve-price curve at a point is concave or convex.

$$R \equiv R(S, q, t; j) \tag{4-1}$$

$$dR = R_t\,dt + R_S dS + \frac{1}{2} R_{SS} (dS)^2 \tag{4-2}$$

Substituting the price process from equation (3-1) into equation (4-2) gives equation (4-3). Note that the change of reserve with time is the depletion rate q.

$$dR = (-q_t + (\alpha - \delta) R_S S + \frac{1}{2} R_{SS} \sigma^2 S^2) dt + R_S \sigma S dz \tag{4-3}$$

The process governing the reserve can thus be described as a Weiner process with drift term η and variance $\sigma^2_{RS}dt$ as shown in equation (4-4).

$$dR = \eta dt + \sigma_{RS} dz \tag{4-4}$$

This variance is proportional to the sensitivity of the reserve-price curve. In general, the shape of the reserve-price curve which is mine specific determines the value of η. If the shape is convex, then the second and third terms of equation (4-3) are always positive given that the expected capital appreciation in price, α, is mostly greater than the convenience yield, δ. In physical terms, η can be interpreted as the effective depletion rate which could be either less or greater than the case when reserve is assumed to be constant[28]. In the case where reserve is finite, the reserve-price curve is upward sloping ($R_s>0$) and reserve has a diminishing marginal sensitivity to price, $R_{ss} < 0$, and $R_s \rightarrow 0$ as price approaches a certain critical price S_c.

where,

$$\left.\begin{aligned} &\eta = -q_t + (\alpha - \delta) R_S S + \frac{1}{2} R_{SS} \sigma^2 S^2 \text{ and} \\ &\sigma_{RS} = R_S \sigma S \end{aligned}\right\} \tag{4-5}$$

$$\lim_{S \to Sc} dR = -q_t dt \tag{4-6}$$

In general, comparing 2DPM to the constant reserve model (CRM), the depletion rate in 2DPM can be described as the sum of two components: a physical component ‘q’ and a ‘virtual’ component which could be positive or negative (determined by the shape of the reserve-price curve). At any reserve state, a negative virtual component denotes an accelerated depletion and earlier mine closure than scheduled mine life. For a given mine, the accelerated depletion is enhanced by higher volatility of the output price. Also if the current reserve state is not the ultimate reserve, then a higher expected return in the output price leads to ‘restock’ of reserve.

---

[28] In the deterministic case, dR=-qdt, and the drift term is –q

4.3 Generalized Mine Value Model

By substituting the price process defined by equation (3-1) and the reserve process of equation (4-3) into equation (3-19) (and noting the conditions of higher order terms of the standard Weiner process in equation (3-5)), the instantaneous change in the market value of the mine can be described by equation (4-7).

$$\begin{aligned} dM = & \left[-qM_R + \frac{1}{2}\sigma^2 S^2 (M_{SS} + 2R_S M_{SR} + R_S^2 M_{RR} + M_R R_{SS}) + M_t \right] dt \\ & + S(M_R R_S + M_S)(\alpha - \delta)dt + S(M_R R_S + M_S)\sigma dz \end{aligned} \tag{4-7}$$

A look at equation (4-7) reveals that the change in mine value is stochastic with an expected term (the terms multiplying dt) and a random term. The uncertainty in the change in mine value is influenced by the same Weiner process that governs the underlying price. The futures price is also affected by this same uncertainty in the spot price, as such, it is possible to create a portfolio consisting of offsetting positions in the mine and futures contract. The gains (losses) on the futures contract as a result of change in prices offset the losses (gains) in the mine.

By considering a portfolio consisting of a long position in the mine and an appropriate number of short position in a futures contract, it is possible to create an asset that is instantaneously risk-free. Under these conditions the appropriate required rate of return on the asset is the risk-free rate in order to avoid arbitrage opportunities. Consider a portfolio consisting of a long position in the mine and Δ number of short position in a futures contract. The value of such a portfolio, P, is given by equation (4-8).

$$P = M - \Delta F \tag{4-8}$$

The change in P, dP, is given by the sum of the capital gains or losses due to the change in the market values of *M* and *F* plus any 'dividend' or cash-flow accruing from holding the two positions. This is shown in equation (4-9)[29].

$$dP = dM - \Delta dF + Adt \quad ; A(S,q) \tag{4-9}$$

A is the net cash-flow accruing from the long position in the mine. This depends on the physical depletion rate, quality of ore produced and production cost as well as commodity price. Substituting equation (4-7) for *dM* and equation (3-3) for *dF* into equation (4-9) and simplifying gives equation (4-10).

---

[29] A typical futures contract is settled daily so that the gains or losses can be realized immediately.

$$dP = (-qM_R + \frac{1}{2}\sigma^2 S^2(M_{SS} + 2R_S M_{SR} + R_S^2 M_{RR} + R_{SS}M_R - \Delta F_{SS}) + M_t - \Delta F_t)dt \\ + (R_S M_R + M_S - \Delta F_S)dS + Adt \qquad (4\text{-}10)$$

By setting Δ (the number of futures contracts) to be equal to $(M_R R_S + M_S)/F_S$, the random component of the change in the value of the portfolio due to the Weiner process is eliminated and equation (4-10) reduces to (4-11).

$$dP = (-qM_R + \frac{1}{2}\sigma^2 S^2(M_{SS} + 2R_S M_{SR} + R_S^2 M_{RR} + R_{SS}M_R - \Delta F_{SS}) + M_t - \Delta F_t)dt \\ + Adt;\ \Delta = (R_S M_R + M_S)/F_S \qquad (4\text{-}11)$$

The reason for setting Δ to its value is that the number of futures contract is selected in order to completely hedge the risk in the portfolio. In equation (4-10), the risk component in the change in the value of the portfolio is the term multiplying *dS*.

The change in the portfolio value is also independent of investor's preferences since the expected return on the commodity price α is eliminated in equation (4-11). The portfolio return must thus be riskless so the appropriate return should be the risk-free rate in order to avoid riskless arbitrage opportunities. Since the entry into a futures contract does not require upfront cash expenditure, the instantaneous return on the portfolio is given by equation (4-12).

$$\frac{dP}{M} = rdt \qquad (4\text{-}12)$$

From the analytical solution of the futures price in equation (3-16), the futures price is linear in S so that $F_{SS}=0$ and $F_t/F_S=-S(r-\delta)$. Using equations (4-11) and (4-12), the equilibrium differential equation that must be satisfied by the value of the mine is given by equation (4-13) subject to the appropriate terminal and boundary conditions.

$$
\begin{aligned}
&(-q+R_S(r-\delta)S+\frac{1}{2}R_{SS}\sigma^2S^2)M_R+\frac{1}{2}(R_S^2\sigma^2S^2M_{RR}+2R_S\sigma^2S^2M_{SR}+\sigma^2S^2M_{SS})\\
&+(r-\delta)SM_S-rM-M_\tau+A=0
\end{aligned}
\tag{4-13}
$$

The boundary and terminal conditions that apply to the mine in each state are discussed in the Sections that follow. The coefficients of the second order terms in equation (4-13) show that the equilibrium PDE is a parabolic differential equation[30].

4.3.1 Dynamic Value of an Operating Mine

The value of a mine depends on calendar time because the input parameters change with time. As such the inflation adjusted value of an operating mine O(S, R) can be described by equation (4-14) subject to the appropriate boundary conditions.

$$
\begin{aligned}
&(-q+R_S(r-\delta)S+\frac{1}{2}R_{SS}\sigma^2S^2)O_R+\frac{1}{2}(R_S^2\sigma^2S^2O_{RR}+2R_S\sigma^2S^2O_{SR}+\sigma^2S^2O_{SS})\\
&+\ (r-\delta)SO_S-rO+A=0
\end{aligned}
\tag{4-14}
$$

---

[30] $(2R_S\sigma^2S^2)^2-4R_S^2(\sigma^2S^2)^2=0$

The relevant boundary conditions are the zero price and zero reserve conditions as well as the high price condition. Other relevant boundary conditions are the value-matching[31] and the "higher order contact" or "smooth pasting"[32] conditions which are necessary to optimally determine the exercise boundaries (Merton, 1973; Dixit and Pindyck, 1994) in the case when the operations have embedded options. These conditions ensure that the decision to switch from one option to the other is done in such a way as to maximise the value of the mine.

$$O(0,R) = 0 \tag{4-15}$$

$$O(S,0;o) = 0 \tag{4-16}$$

$$O_R(S,R(S);o) = 0 \tag{4-17}$$

$$O_{SS}(S,R;o) = 0; \quad S \to \infty \tag{4-18}$$

Equation (4-15) means that if the commodity price drops to zero at any time, the value of the mine is zero. Equation (4-16) states that, if the reserve drops to zero at any time, the value of the mine is zero. Equation (4-17) means that, at a given price, the mine value does not change as a result of changes in reserve when the reserve is at the level determined by that price. Equation (4-18) states that the mine value is linear in price at extreme commodity prices, consistent with linear cash-flow model (Laughton and Jacoby, 1991). This condition is also equivalent to saying that when prices get very large the mine will operate under the same policy until complete exhaustion so that flexibility has no additional value.

---

[31] Value-matching because the value of the unknown function is set equal (matched) to the value of the known terminal payoff.

[32] Smooth pasting condition is used to determine the optimal exercise boundary. This is a general condition which requires that the functional form of the two dependent variables should meet tangentially at this boundary. Intuitively, the objective is to maximize the payoff on exercise of the option so the smooth pasting condition is equivalent to differentiating the payoff function and setting the results to zero in a fashion consistent with optimization in calculus.

When an operating mine has the flexibility to temporarily close down then equation (4-15) can be replaced with equations (4-19) and (4-20) which relates the value of an open mine *O* to that of a closed mine *C*.

$$O(S_1^*,R;o) = \max(C(S_1^*,R;c) - Kc, 0) \tag{4-19}$$

$$O_S(S_1^*,R;o) = C_S(S_1^*,R;c) \tag{4-20}$$

Kc is the cost incurred by bringing the mine to a temporary closure.

When equation (4-14) is maximized subject to all the conditions, critical prices, S_1* and S_2* can be determined (Cortazar and Schwartz (1997); Frimpong (1992)) that optimize a given operating policy of the mine. For prices above S_2*, the mine is operated up to the end of mine life and a closed mine is reopened by incurring a reopening cost (see value of a closed mine in next section). For prices between S_1* and S_2* a closed mine remains closed and an operating mine remains opened. It may be optimal to abandon the mine for prices below S_1*.

Equation (4-19) and (4-20) are the value-matching and smooth pasting conditions respectively. The value-matching condition states that, at any time during the operating stage, the operations can be terminated by considering both the closure and abandonment options. One has to evaluate if the mine should continue to operate or it is optimal to exercise the close option and receive the value of a closed mine or exercise the abandonment option and incur the cost of abandonment. The smooth pasting condition ensures that the decision to switch from open to close is made in such a way to maximize the value of the mine (Dixit and Pindyck, 1994).

4.3.2 Dynamic Value of a Closed Mine

For a closed mine the production rate is zero, and thus, no positive income from operations. The principal difference between a closed mine and an abandoned mine is that for a temporarily closed mine, there is a periodic maintenance cost which ensures that re-opening is possible when conditions become favourable. The equilibrium equation for the value of the closed mine C(S, R; c) is therefore given by equation (4-21).

$$\begin{aligned}&(R_S(r-\delta)S+\frac{1}{2}R_{SS}\sigma^2S^2)C_R+\frac{1}{2}(R_S^2\sigma^2S^2C_{RR}+2R_S\sigma^2S^2C_{SR}+\sigma^2S^2C_{SS})\\&+\ (r-\delta)SC_S-rC+Ac=0\end{aligned}\qquad(4\text{-}21)$$

Ac is the cost of maintaining the temporary closed mine.

For a closed mine that has the possibility of re-opening, there are two free boundaries that have to be optimally determined as part of the solution. An upper boundary which determines the critical price at which the closed mine should be re-opened and a lower boundary that determines the critical price at which the closed mine should be abandoned. There are thus two value-matching and smooth pasting conditions.

$$C(S,0;c)=0 \tag{4-22}$$

$$C(S_0^*,R;c)=0 \tag{4-23}$$

$$C_S(S_0^*,R;c)=0 \tag{4-24}$$

Equation (4-22) is the zero reserve condition. Equation (4-23) means that, at a certain critical price, a closed mine should be abandoned at a net cost of zero and Equation (4-24) is the smooth pasting conditions that ensures that the critical price S^*_0 is such as to maximize the value of the mine.

$$C_R(S,R(S);c)=0 \tag{4-25}$$

$$C(S_2^*,R;c)=O(S_2^*,R;o)-Kco \tag{4-26}$$

$$C_S(S_2^*,R,\tau_c;c)=O_S(S_2^*,R,\tau_c;o) \tag{4-27}$$

Equation (4-25) means that at a given price, the value of the closed mine does not change with respect to reserve when the reserve is at maximum corresponding to that spot price. Equation (4-26) states that, at a critical price, a closed mine may be re-opened to receive an operating mine by incurring a re-opening cost of Kco. Equation (4-27) is the smooth pasting condition.

4.3.3 Dynamic Value at Mine Development Stage

Similarly, the value of the mine at the development stage, assuming that all development expense is at the beginning of development, satisfies equation (4-28).

$$\begin{aligned}&(R_S(r-\delta)S+\frac{1}{2}R_{SS}\sigma^2S^2)D_R+\frac{1}{2}(R_S^2\sigma^2S^2D_{RR}+2R_S\sigma^2S^2D_{SR}+\sigma^2S^2D_{SS})\\&+\ (r-\delta)SD_S-rD-D_{\tau d}=0\end{aligned} \qquad (4\text{-}28)$$

At the end of development, the value of the mine should be equal to the greater of the value of an opened or closed mine. However, if the decision to invest is done optimally, the value after development will always equal that of an opened mine.

$$D(S,R,T_d;d)=\max(O(S,R;o),C(S,R;c)) \qquad (4\text{-}29)$$

$$D(S_{min},R(S_{min}),\tau_d;d)=0 \qquad (4\text{-}30)$$

$$D(S,R(S_{min}),\tau_d;d)=0 \qquad (4\text{-}31)$$

Equation (4-29) implies that, mine value at the end of development is equal to the maximum of the value of the opened or closed mine. If at the end of development, prices are relatively small then it may be optimal to close the mine instead of to operate it. Equations (4-30) and (4-31) are the lower price and reserve conditions respectively.

$$D_R(S,R(S),\tau_d;d)=0 \qquad (4\text{-}32)$$

$$D_{SS}(S,R,\tau_d;d)=0\ ;\quad S\to\infty \tag{4-33}$$

Equation (4-32) states that, at a given price, the value of the developing mine does not change with respect to reserve when the reserve is at maximum corresponding to that spot price. Equation (4-33) states that the value of a mine under development is linear in price at extreme commodity prices.

4.3.4 Dynamic Value at the Investment Decision Stage

The investment decision constitutes the choice to exercise the option to invest by paying for the cost of development and in return receive a developed mine with all the embedded options. The value of the mine at this stage satisfies equation (4-34) subject to the terminal and boundary conditions in equations (4-35) to (4-40).

$$\begin{aligned}&(R_S(r-\delta)S+\frac{1}{2}R_{SS}\sigma^2S^2)V_R+\frac{1}{2}(R_S^2\sigma^2S^2V_{RR}+2R_S\sigma^2S^2V_{SR}+\sigma^2S^2V_{SS})\\&+\ (r-\delta)SV_S-rV-V_{\tau u}=0\end{aligned} \tag{4-34}$$

Whilst the traditional NPV suggests one should invest in a project if the NPV is positive, here it is necessary to determine the critical price above which it is optimal to invest. This results from the fact that the investment decision is no longer treated as a now or never decision since there is the option to wait on the decision. This critical price is also determined using the smooth pasting and value-matching conditions.

$$V(S,R,T_u;u)=\max(D(S,R,\tau_d;d)-Io,0) \tag{4-35}$$

Equation (4-35) states that when the available time to start development is exhausted, the value of the undeveloped mine depends on whether it is optimal to pay the development cost and receive a developed mine or give the concession up.

$$V(S,R(S_{min}),\tau_u;u)=0 \tag{4-36}$$

$$V(S_{min},R(S_{min}),\tau_u;u)=0 \tag{4-37}$$

$$V_R(S,R(S),\tau_u;u)=0 \tag{4-38}$$

$$V(S_U^*,R,\tau_u;u)= D(S_U^*,R,\tau_d;d)-Io \tag{4-39}$$

Equations (4-36) and (4-37) are the low reserve and low price conditions. Equation (4-39) means that, at a critical price, S_U^*, during the lease period, the option to invest could be exercised by paying the investment cost and receiving a developed mine in return.

$$V_S(S_U^*,R,\tau;u)= D_S(S_U^*,R,\tau;d) \tag{4-40}$$

Equation (4-40) is the smooth pasting condition which ensures that the critical price is optimally selected. S_U^*, has to be determined as part of the solution since it is not known a priori.

Equivalent model equations in which reserve is characterised as known and constant will be similar to the ones developed in Section 4.3 with all the mixed derivative terms, as well as, the Rs and Rss terms set equal to zero. The equations above do not have a tractable analytical solution and so they have to be solved numerically.

4.4 Case Study of the 2D CTSP Price Model

The example below is used to illustrate the nature of the solution provided by the model. A hypothetical mine is considered with a maximum potential inventory of 127.8 million tonnes if the mill cut-off grade is 0.23 % of copper (at copper price of $1.8/lb).

Table 4-1: Input Data to Model 2DPM

Ultimate Reserve	127.8mt
Ultimate average grade; AG	0.65%
Recovery efficiency	80%
Cost per output of copper, C	$0.65/lb
Extraction rate, q	5mt/year
Real risk-free interest rate; r	6%
Convenience yield δ	4%
Output price volatility σ	20%
Maintenance cost, Ac	$0.5m/year
Closure Cost, Kc	$20m
Re-Open Cost , Kco	$20m
Investment Cost, Io	$150m

Table 4-1 shows other project parameters. The convenience yield, risk-free rate (inflation adjusted) and price volatility are adapted from Dixit and Pindyck (1994, Chapter 7). All cost data in Table 1 are constant dollars. Other inputs to the model include the tonnage-average grade curve, reserve-price and reserve-price sensitivity curves are shown in Figures 4-1, 4-2 and 4-3 respectively.The grade-tonnage curve in Figure 4-1 is the results from a typical ordinary kriging (or can be obtained from any other geostatistical method to estimate reserve). The price-reserve in curve in Figure 4-2 is generated by determining the break-even price corresponding to the cut-off grade. Given the cost/ton and the recovery efficiency one can easily determine the break-even price corresponding to given cut-off grade by using equation (4-41).

$$P = \frac{10000 \times \text{cost}(\$/t)}{2204 \times \text{cutoff}(\%) \times \text{efficiency}(\%)} \tag{4-41}$$

For example at a cut-off of 0.23%Cu at an efficiency of 80% the break-even price is $1.83/lb if cost is $7.45/t of ore (0.65/lb of copper).

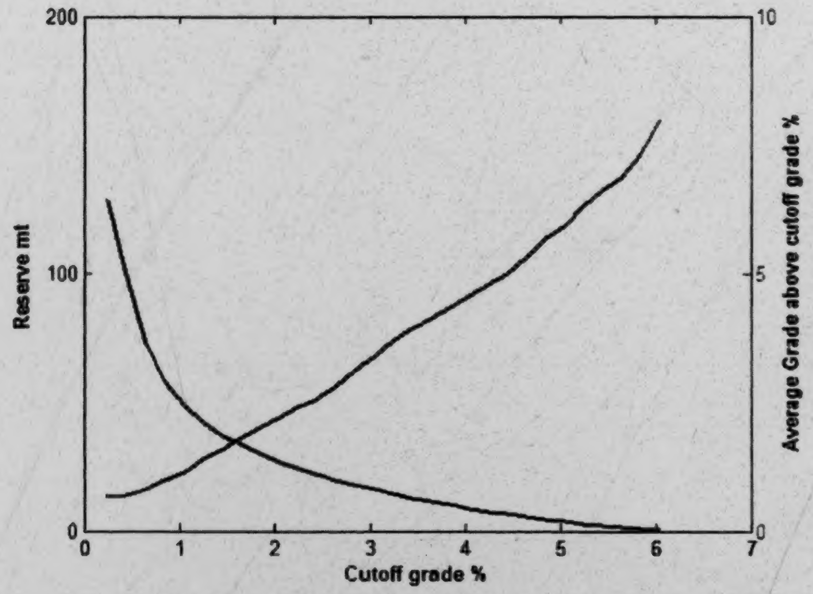

Figure 4-1: Grade-tonnage curve for hypothetical mine

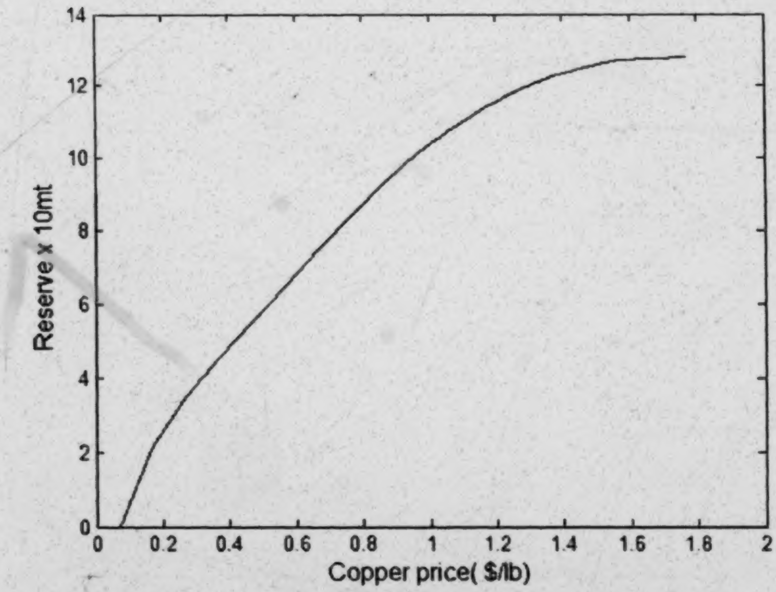

Figure 4-2: Reserve-Output price curve for hypothetical mine

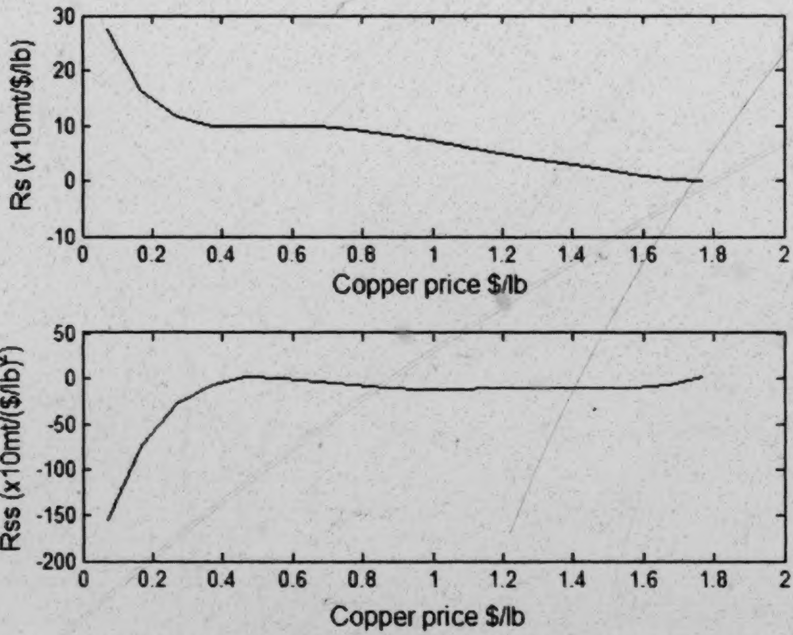

Figure 4-3: Reserve- Price Sensitivity Curves

At an annual production rate of 5mt, the mine has a potential mine life of about 25 years if output prices are very favourable. The mine could however be abandoned anytime at a net cost of zero to the operator. At unfavourable prices, (price less than $1.8/lb), the mine life could be less than this since economically mineable reserves could be reduced significantly. Development is assumed to be instantaneous and the option to invest is assumed to be perpetual[33]. The ability to be able to capture possible changes in reserve during valuation is the novelty of this model. In solving this as a boundary value problem, the reserve boundary is a curvilinear boundary, which is fully defined by the reserve-price curve up to a certain critical price.

4.5 Solution Procedure

The developed equations do not have closed form analytical solutions and numerical methods must be employed in their solution. The equations are solved using MATLAB and FEMLAB's equation-based modeling, an interactive environment to model single and coupled phenomena based on partial differential equations (PDEs). FEMLAB employs the finite element method (FEM)[34] and offers a complete multiphysics modeling environment where one can simultaneously solve any combination of coupled partial differential equations. This modeling capability is useful since the mine is governed by a different partial differential equation in the closed, open, development and investment stages which are couple at the free boundaries. Modeling in FEMLAB involves geometric modeling, specification of model equations and boundary conditions; generation of unstructured meshed (using an automatic mesh generator based on the Delaunay triangulation algorithm in MATLAB), numerical integration of the model equations and boundary conditions and post processing. Finite element method was selected as the solution procedure because of its ability to easily deal with arbitrary geometries. In the 2DPM model, the reserve-price curve constitutes a boundary of the solution domain. This boundary is therefore curvilinear and the finite element method facilitates the descretization of geometries with such boundaries. The choice of FEMLAB also facilitates the solution process in the sense that one does not have to write a computer code to descretize both the governing equation and the solution domain. FEMLAB has the capability to automatically deal with these details. It also seamlessly works with MATLAB so it is possible to tap into the rich capabilities of MATLAB in numerical modeling whilst taking advantage of the finite element modeling capabilities of FEMLAB.

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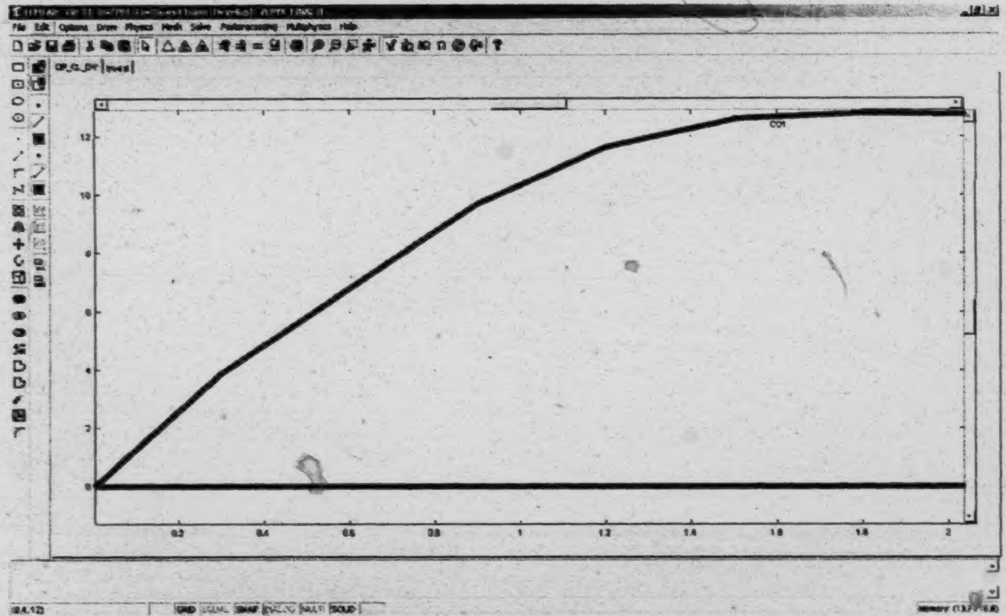
[33] In many cases the threshold price at which it is optimal to investment is not very sensitive to lease periods more than two years so it is reasonable to assume perpetuity since most mineral leases are for periods more than two years. See Dixit and Pindyck (1995) and Paddock et al 1988.

[34] See section B4 in Appendix B for general overview of FEM

FEMLAB is used in conjunction with MATLAB to derive the optimal exercise boundaries by using the steepest descent optimization algorithm. The reserve-price sensitivity curves - *Rs* and *Rss* - are generated from the reserve-price curve using *cftool,* a curve fitting tool in MATLAB. The data-set so generated in MATLAB for *Rs* and *Rss* are used to define an interpolation function in FEMLAB.

4.5.1 *Geometry Modeling in FEMLAB*

To specify a model in FEMLAB, one has to define the geometry or the domain of interest. The domain makes up the bounded region of space in which one solves the governing equation(s). FEMLAB offers the flexibility to divide the domain into sub-domains and this is especially useful if the different sections of the domain are governed by entirely different physics. In this case, in addition to the external boundaries of the domain, boundary conditions must be specified on the interior boundaries. It is also possible to have a multiple geometry model in which case each 'sub-domain' is defined as a unique geometry on its own. A combination of these two capabilities is used in defining the model geometry.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Multiphysics
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10
8
6
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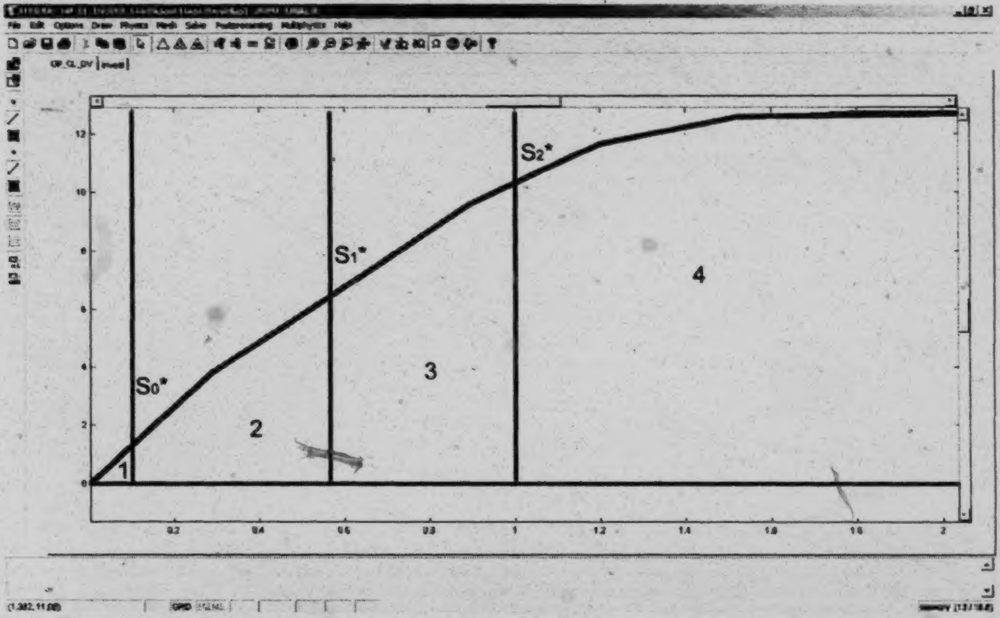
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
GRID
EQUAL
SNAP
MULTI
SOLID

Figure 4-4: Basic Geometry for modeling equilibrium equations.

A 2-dimensional geometry with reserve and price as the state variables is generated. The left-most boundary is curvilinear and is defined by using the price-reserve curve up to a certain critical price as shown in Figure 4-4. This geometry is divided into four sub-domains by three interior boundaries as shown in Figure 4-5.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S0*
S1*
S2*
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2
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0
2
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6
8
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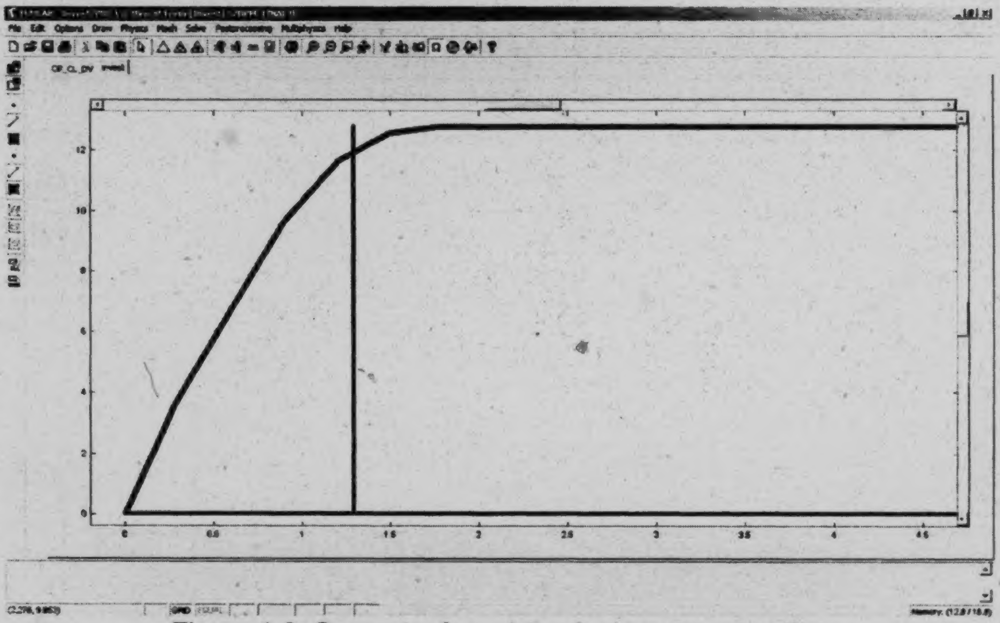
12
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 4-5: Geometry for solving Closed, Open and Development Options

These interior boundaries define the abandonment, closure and reopen thresholds and are arbitrarily selected such that abandonment < closure < re-open. The resulting geometry in Figure 4-5 is used as the relevant geometry for the solution of the equilibrium equations for open, close and development options. The close option is solved in the subdomains that are marked *2* and *3* and the open option is solved in *3 and 4*. The development option is however solved in subdomains *2*, *3* and *4*.

A second geometry based on Figure 4-4 is generated for the solution of the investment option. But in this geometry only one interior boundary (which represents the investment threshold) is used to divide the domain into invest and wait sub-domains as shown in Figure 4-6. The geometric model thus consists of two separate geometries each with sub-domains.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8
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12
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1
1.5
2
2.5
3
3.5
4
4.5

Figure 4-6: Geometry for solving for investment option

4.5.2 *Specifying PDEs in FEMLAB*

The equation that describes an open mine in equation (4-14) is used to demonstrate how the equations are specified using the coefficient form of FEMLABS's equation based modeling. The coefficient form is appropriate for linear or almost linear system of equations. Equation (4-14) is linear in the PDE but the coupling with equation (4-21) - the equation that characterizes the close option - through the boundary conditions of equations (4-19) and (4-20) introduce some non-linearity[35]. In the coefficient form, if *u* is a dependent variable the general PDE problem is defined in FEMLAB as in equation (4-42).

$$\begin{cases} d_a \dfrac{\partial u}{\partial t} + \nabla\cdot(-c\nabla u - \alpha u + \gamma) + \beta\cdot\nabla u + au = f & \text{in } \Omega \\ \mathbf{n}\cdot(c\nabla u + \alpha u - \gamma) + q^* u = g - h^T \mu & \text{on } \partial\Omega \\ hu = r & \text{on } \partial\Omega \end{cases} \quad (4\text{-}42)$$

where

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\ldots \frac{\partial}{\partial x_n} \right) \quad (4\text{-}43)$$

The first equation in equation (4-42) above is the PDE, and it must be satisfied in the domain. The second and third equations are the boundary conditions, and they must be satisfied on the boundaries. The second equation[36] is a *generalized Neumann* boundary condition, and the third equation is a *Dirichlet* boundary condition. ∇ is the gradient vector operator defined as shown in equation (4-43), Ω is the computational domain - the union of all subdomains - $\partial\Omega$ is the domain boundary and **n** is the outward unit normal vector on the boundary. The variable

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[35] This non-linearity requires the use of the non-linear solver in solving the resulting system of equations.

[36] In FEMLAB q^* is in fact designated as q, but q^* is adopted to differentiate it from the production rate q

μ is an unknown function known as the *Lagrange* multiplier[37] and τ denotes transpose. All other terms in equation (4-42) are coefficients and are scalars except α, β and γ which are vectors of dimension *k*. Depending on the form of the equation, the coefficient *c* can however be a *k x k* matrix, where *k* is the number of independent variables.

All PDEs and the associated boundary conditions are written in the form of equation (4-42). Equation (4-14) is re-written as equation (4-44). This equation is made active in subdomains 3 and 4 in Figure 4-5.

$$-\frac{1}{2}\nabla\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\nabla \mathrm{O} + \mathrm{rO} + \begin{pmatrix} \beta_{\mathrm{S}} \\ \beta_{\mathrm{R}} \end{pmatrix}\nabla \mathrm{O} = \mathrm{A} \tag{4-44}$$

where

$$\begin{cases} c_{11} = \sigma^2 S^2; \quad c_{12} = \mathrm{R_S}\sigma^2 S^2; \quad c_{21} = \mathrm{R_S}\sigma^2 S^2; \quad c_{22} = (\mathrm{R_S}\sigma)^2 S^2 \\ \beta_S = -(\mathrm{r} - \delta - \sigma^2)\mathrm{S} \\ \beta_R = \mathrm{q} - \frac{1}{2}\mathrm{R_{SS}}\sigma^2 S^2 - (\mathrm{r} - \delta)\mathrm{R_S} S + \mathrm{R_S}\sigma^2 S \end{cases} \tag{4-45}$$

4.5.3 Specifying Boundary Conditions

The appropriate boundary conditions for equation (4-14) are equations (4-16) to (4-20).

---

[37] This is an extra dependent variable introduced in the Neumann boundary condition when a constraint (Dirichlet condition) is added at a boundary. See the Section "What Equations Does FEMLAB Solve?" in FEMLAB Documentation for further discussion on the Lagrange multiplier.

Equation (4-16) is implemented as a *Dirichlet* condition with the coefficients $\boldsymbol{h}=1$, $\boldsymbol{r}=0$, $\boldsymbol{q^*}=0$ and $\boldsymbol{g}=0$.

Equations (4-19) and (4-20) are implemented together as a *Dirichlet* condition with the coefficients $\boldsymbol{h}=1$, $\boldsymbol{r}=\max(C-Kc,0)$, $\boldsymbol{q^*}=0$ and $\boldsymbol{g}=\frac{n_S c_{11} SC_S + n_S c_{12} O_R}{2}$. The *Dirichlet* condition in FEMLAB also has a *Neuman* condition. This condition is specified on the boundary marked $S_1{}^*$ in Figure 4-5.

Equation (4-17) is implemented as a *Neuman* condition with the coefficients $\boldsymbol{q^*}=0$ and $\boldsymbol{g}=\frac{n_R c_{21} O_S + n_S c_{11} O_S}{2}$; n_R and n_S are the components of the unit vector normal to the boundary. Equation (4-18) is implemented as a generalized *Neuman* condition. To implement this, O_{SS}, R_S and R_{SS} in equation (4-14) are set to zero. This results in equation (4-46). This is implemented by setting the coefficients in the *Neuman* condition to the appropriate values as explained below. Implementing the *Neuman* condition means implementing equation (4-47), which can be expanded to give equation (4-48) by substituting $\mathbf{n_s}=1$ and $\mathbf{n_R}=0$ at the boundary.

$$-qO_R + (r-\delta)SO_S - rO + A = 0 \tag{4-46}$$

$$\frac{1}{2}(\mathbf{n_s}\ \mathbf{n_R})\begin{pmatrix}\sigma^2 S^2 & 0\\ 0 & 0\end{pmatrix}\nabla O + q^* O = g \tag{4-47}$$

$$\frac{1}{2}\sigma^2 S^2 O_S + q^* O = g \tag{4-48}$$

Equation (4-46) can be written in the form of (4-49) in order to be in the same form as equation (4-48).

$$\frac{1}{2}\sigma^2 S^2 O_S - \frac{\sigma^2 S^2 r}{2S(r-\delta)} O = \frac{-A\sigma^2 S^2}{2S(r-\delta)} + \frac{\sigma^2 S^2}{2S(r-\delta)} q O_R \qquad (4\text{-}49)$$

Comparing equation (4-49) and (4-48) and comparing coefficients means setting $\boldsymbol{q^*} = \frac{-rc_{11}}{2S(r-\delta)}$ and $\boldsymbol{g} = \frac{-Ac_{11}}{2S(r-\delta)}$. In order to fully implement the boundary condition in (4-49) there is also the introduction of a weak term given by $\frac{\sigma^2 S^2 q O_R O\_test}{2S(r-\delta)}$. The weak term contribution facilitates the modeling of a PDE on a boundary[38]. All the other equilibrium equations that describe the different states of the mine and their associated boundary conditions are formulated as described in Section 4.5.3.

4.5.4 Optimization

The equilibrium PDEs are coupled at the boundaries and so they have to be solved simultaneously. The multiphysics capability of FEMLAB is able to seamlessly implement the simultaneous solution of the equations. To facilitate solution and convergence, the closed and open options are solved first since the coupling is stronger. The solutions obtained are not necessarily the optimal solution since the threshold boundaries, abandon, close, reopen were arbitrarily chosen albeit to conform to a particular order. For an optimal solution however, these free boundaries must be uniquely determined if an optimum exists. The smooth pasting conditions require that for optimal solution the derivatives of the closed and open mine with respect to price must be equal, thus $O_S(S_1^*, R; o) = C_S(S_1^*, R; c)$ on the closed boundary, $O_S(S_2^*, R; o) = C_S(S_2^*, R; c)$ on the re-open boundary and $C_S(S_0^*, R; c) = 0$ on the abandonment boundary. The derivatives at the exercise boundaries are compared after the initial solution and the boundaries are adjusted using the steepest descent algorithm with constant step size as shown in equation (4-50).

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[38] See Section B6 in Appendix B for further implementation details.

$$\begin{aligned} So^{*(k+1)} &= So^{*(k)} - \alpha_1 C_S(R(So^{*(k)}) \\ S1^{*(k+1)} &= S1^{*(k)} - \alpha_2 \left[O_S(R(S1^{*(k)}) - C_S(R(S1^{*(k)}) \right] \\ S2^{*(k+1)} &= S2^{*(k)} + \alpha_3 \left[O_S(R(S2^{*(k)}) - C_S(R(S2^{*(k)}) \right] \end{aligned} \tag{4-50}$$

This algorithm is chosen purely because of the ease of implementation. The αs in equation (4-50) are the different step sizes applied at each boundary. The value of the constant step size that guarantees convergence is given by equation (4-51) where $\lambda_{max}(Q)$ denotes the largest eigenvalue of the symmetric matrix generated from the function that is being optimised.

$$0 < \alpha_i \leq \frac{2}{\lambda_{max}(Q)} \tag{4-51}$$

Although FEMLAB has the capability to numerically solve for the eigenvalues using the *eigenvalue solver*, a heuristic approach is used in selecting the αs. This is done in order to reduce the solution time. The αs are selected in such a way that the first adjustment is not more than 10% of the initial guess for the boundaries as shown in equation (4-52).

$$\alpha_i = \frac{0.1 S_i^{*(1)}}{\left| O_s^{(1)}(R(S_i^{*(1)})) - C_s^{(1)}(R(S_i^{*(1)})) \right|} \quad \forall i = 0,1,2 \qquad (4\text{-}52)$$

Since the price of copper is single digit any adjustment that does not change the boundary by up to one half of a cent is considered insignificant. Therefore the convergence criterion used is that shown in equation (4-53). This condition must simultaneously be met at all boundaries for the solution to be deemed as having converged.

$$\left| S_i^{*(k+1)} - S_i^{*(k)} \right| < \varepsilon \quad \forall i = 0,1,2; \quad \varepsilon = 0.005 \qquad (4\text{-}53)$$

With an optimal solution of the closed and open options, the development option is then solved. An *extrusion coupling variable*[39] that is defined as the maximum of the closed and open solution is generated during the solution of the closed and open option and used in the solution of the development option.

The investment option is then solved by using another *coupling variable* defined during the solution of the development option. To determine the optimal investment threshold, the steepest descent algorithm described above is employed again. To facilitate convergence the initial guess for the investment threshold boundary is set equal to the optimal re-open boundary obtained in the solution of the open and closed options. This only makes sense since for reasonable model parameters the overall cost incurred by a closed mine to reopen will be lower than that of a new mine. The trigger price to reopen a closed mine should therefore be lower than that for investing in a new mine.

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[39] Coupling variables are used in FEMLAB to make the value of an expression available nonlocally.

4.6 Discussion of Results

Figure 4-7 shows results using 2DPM to value a mine with potential reserve of 127.8mt. Compared with the CRM model that uses this potential level of reserve as its basis, it is clear why the value under constant cut-off grade policy and hence constant reserve has a higher value.

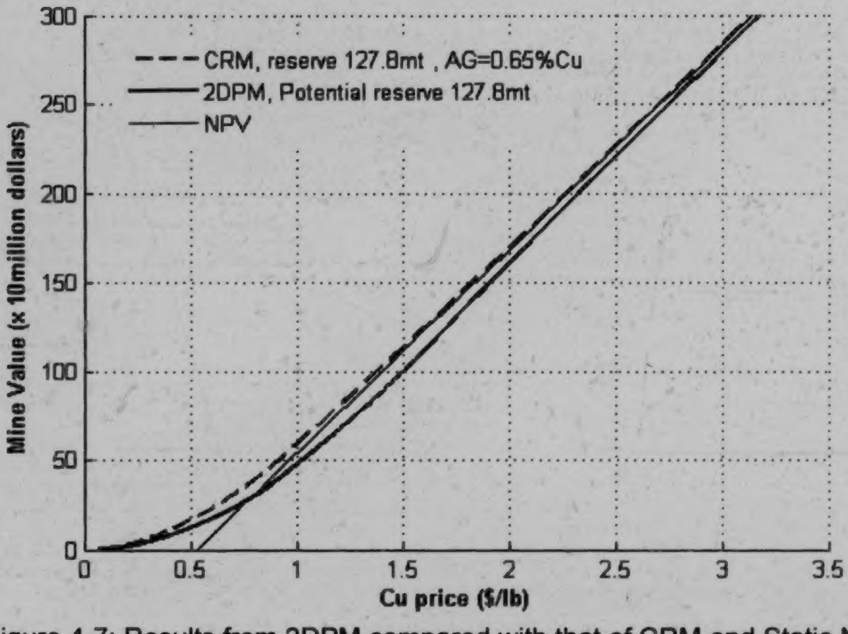

Figure 4-7:.Results from 2DPM compared with that of CRM and Static NPV

At copper price of $1.5/lb, a 127.8mt reserve at a cut-off grade of 0.23%Cu (average grade of 0.65%), is valued at $1,134million by CRM. 2DPM values the mine considering this level of reserve as potential at $996million. This stems from the fact that, in the price sensitive region 2DPM accounts for the possibility of 'virtual depletion' that may result from change in operating cut-off grade. As prices get relatively large well above the price sensitive region, the value from the 2DPM model approaches that from CRM.

At low prices, 'effective' depletion is higher when using 2DPM. If there is not an equivalent increase in cash-flow from the 'additional' depletion then the value of the resulting mine should be lower if compared with the case of constant reserve. However if 'virtual' depletion (by increasing cut-off grade) has the effect of increasing cash-flow[40] then the effect of reserve variability from cut-off grade flexibility is mixed for model parameters.

For a given model parameters, the value of a mine with constant level of reserve and fixed cut-off grade has a higher value if compared with the equivalent model that recognises this level of reserve as 'potential' reserve[41]. The intuition here is clear since a mine that has proven reserves of 150mt of ore has more value than that which has probable or possible reserves of 150mt. The difference in value depends on the extent to which grade affects reserve size and the nature of the reserve-price curve; the steeper the curve the lower the difference.

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[40] Cash-flow can be increased through the increase in average quality of the periodic production.
[41] This level of reserve is realized if output price reaches a certain critical level.

In Figure 4-7, the results from the NPV approach is also compared with that from 2DPM. It is seen that generally beyond the region where there is the flexibility to exercise the close option (in this case price beyond 0.782 cents)[42] the NPV approach over-estimates the value of the asset. This is also due to the constant reserve assumption used in calculating the NPV. At a price of $0.7/lb NPV values the asset at $194million dollars whilst 2DPM values it at $235million dollars but at a price of $1.0/lb NPV values it at $538.2million whilst 2DPM values it at $475.5million.

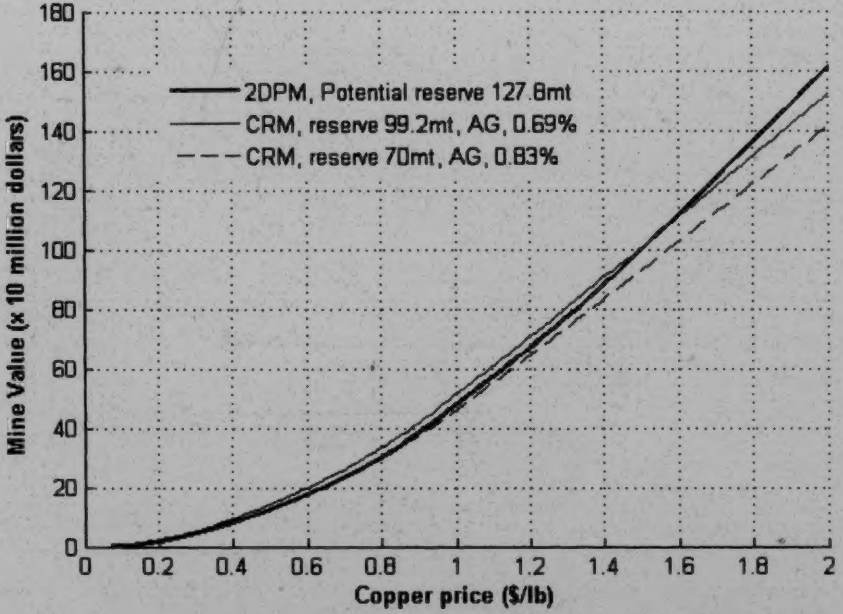

Figure 4-8:.2DPM compared with CRM in Deposit with low grade variability.

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[42] In general price slightly above marginal cost

On the other hand, a project that is valued at its current level of reserve (with level less than the ultimate reserve) using a fixed cut-off grade without consideration of its potential reserve may be under-valued. This is especially so if the deposit has low variability in grade. In this case, the tonnage-grade curve is very steep so that the 'quantity effect' on value is more dominant over the 'quality effect'. This will be the case for massive low-grade deposits where there is generally low variability in grade. This situation is shown in Figure 4-8. For the tonnage-grade curve used in this model, operating at an average grade of 0.83% copper leaves reserves at a level of 70mt. CRM model values this mine at $928million at a copper price of $1.5/lb compared to $996 million by 2DPM. For vein type deposits or deposits with high variability in grade however, CRM values may even be significantly higher than that of 2DPM even if the level of reserve used in CRM valuation is lower. This is expected if coupled with higher quality of ore, the current reserve level is also considerably high. In Figure 4-8, at a copper price of $1.4/lb and operating at a constant average grade of 0.69% of copper (with a level of reserve of 99.2mt), CRM values this mine at $905million[43] while 2DPM values this mine at $879million operating with cut-off grade flexibility and a potential reserve of 127.8mt.

Table 4-2: Exercise Boundaries for Commodity Price ($/lb); CRM and 2DPM

	2DPM	CRM
Abandon threshold, S_0*	0.0721	0.0561
Close threshold, S_1*	0.5843	0.5633
Re-open threshold, S_2*	1.0991	1.1048
Investment threshold, S_u*	1.4092	1.2922

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[43] This value will even be higher if there was higher grade variability.

Table 4-2 shows the exercise boundaries under both CRM and 2DPM. The price at which the mine is abandoned is about 7 cents/ lb using 2DPM and 5.5 cents/lb in using CRM[44]. Since the mine value is lower in 2DPM (both operating mine and closed mine as shown in Figure 4-9, at the same maintenance cost, a mine will be abandoned earlier under the 2DPM than CRM.

For the closed and re-open boundaries 2DPM predicts higher exercise boundary for close (0.5843 vs. 0.5633) but lower value for open (1.0991 vs. 1.1048) than CRM. This difference is marginal and may not be conclusive that this is always the case.

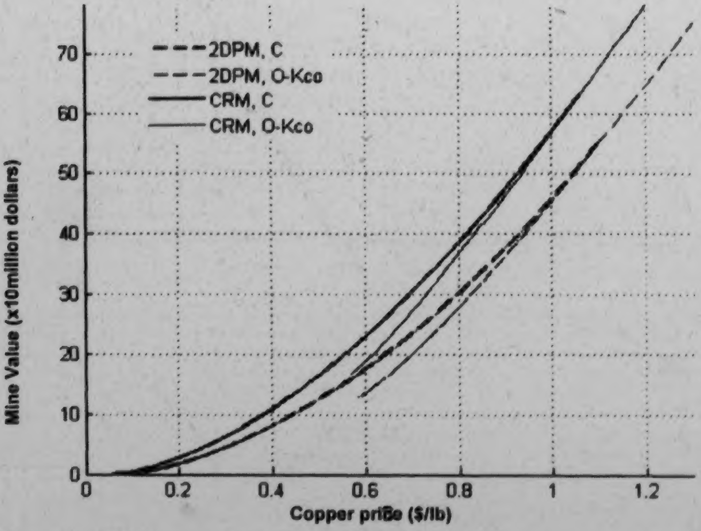

Figure 4-9: Value of Closed and Open Mine as a function of price

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[44] The abandonment price is low with respect to the cost/lb. This low value is however realistic since for a mine with no maintenance cost the theoretical abandonment price is zero. For a mine with maintenance cost however, the abandonment price depends on the maintenance cost as against the value of the option to reopen the mine. Lower maintenance cost leads to late closure and higher option value also lead to late closure. See Chapter 5 on the discussion of abandonment price and size of mine.

Figure 4-10 shows the relative values of open to close for both CRM and 2DPM. The decision to switch from open to close depends on the relative values of the open and close options, as well as, the switching cost. At the same switching cost the ratio is lower for 2DPM at lower prices, meaning close is relatively more attractive in 2DPM than in CRM so the mine is closed earlier. At higher prices however, the ratio is higher for 2DPM meaning the open option is more attractive so a closed mine gets reopened earlier using 2DPM than using CRM.

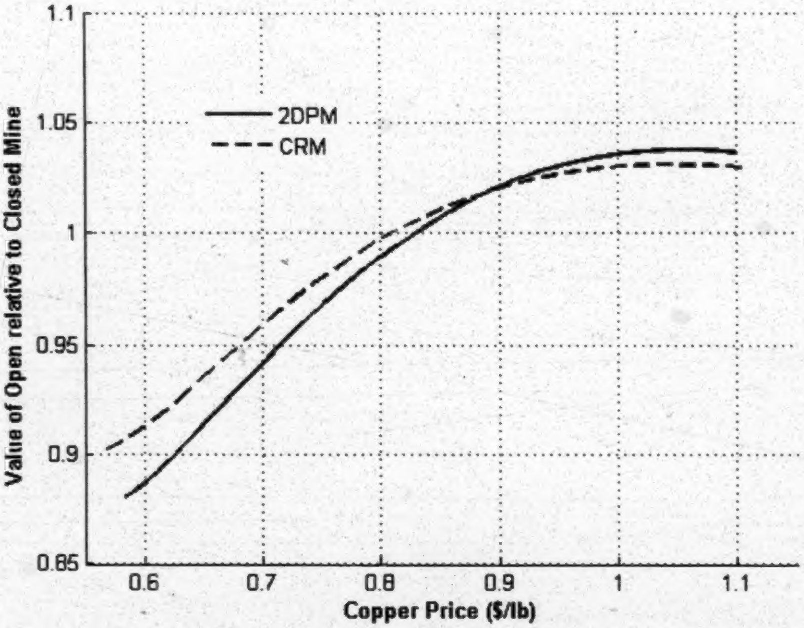

Figure 4-10: Value of an Open Mine Relative to a Closed Mine

2DPM predicts higher critical price (1.41 cents vs. 1.28 cents) at which investment should be made. Both the value of a developed mine and the value of the option to invest are lower for 2DPM than that from CRM as show in Figure 4-11. Higher critical investment price makes sense since for the same investment dollars, prices have to be high enough to offset any risk due to lower mine values. 2DPM thus predicts longer waiting time by firms. Although longer waiting times mean less investment under the 2DPM, firms have the opportunity to invest under more favourable conditions and be subject to less risk due to both price and reserve variability.

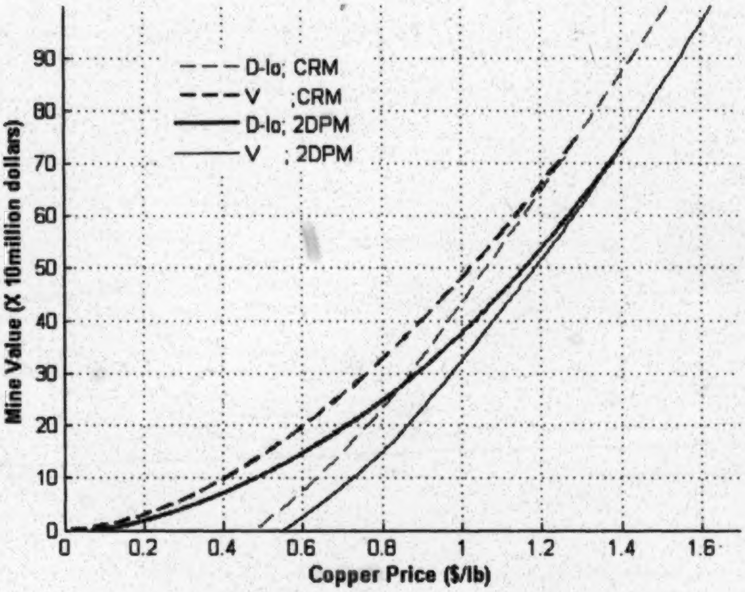

Figure 4-11: Value of the option to invest

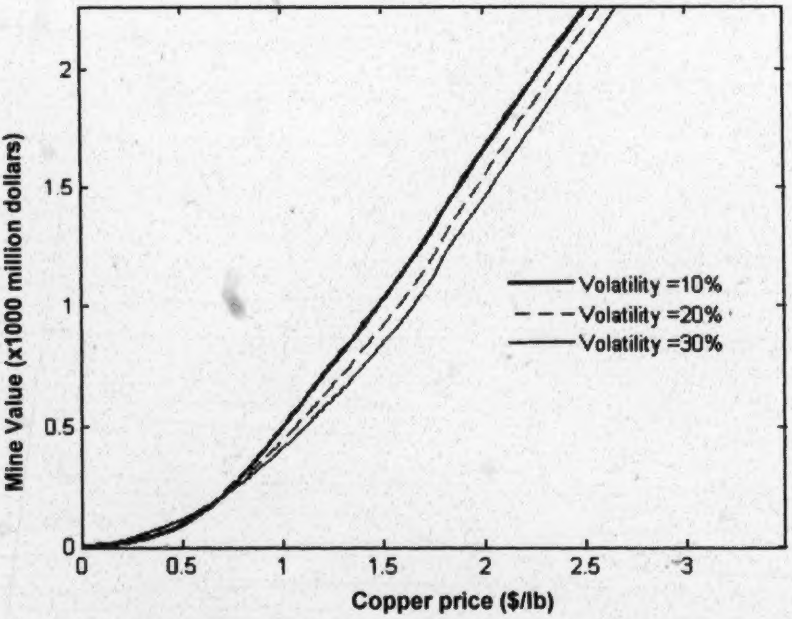

Figure 4-12: Value of Mine with Varying Volatility in Cu price; 2DPM.

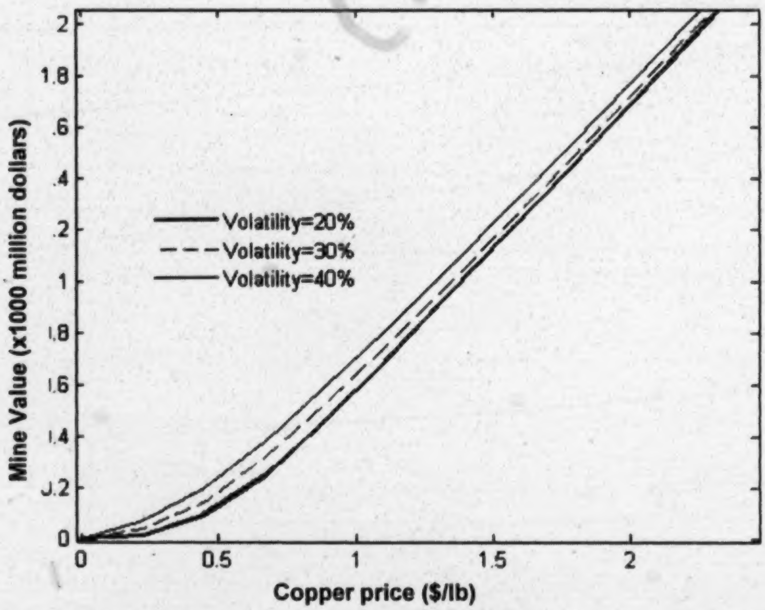

Figure 4-13: Value of Mine for Varying Volatilities in Cu Price; CRM

In the 2DPM however, increase in volatility of commodity price generally leads to a decrease in value of the mine. This might seem counter intuitive in the first place. This intriguing result stems from the fact that for a finite reserve space, with reserve having a diminishing marginal sensitivity to price, there is a limit on the upside potential of how reserve evolves under price swings. In this case the concept that an increase in volatility increases the value of an option because there is unlimited upward potential does not strictly apply.

Although there is also a floor (zero) on the downside, reserve is more sensitive to price swings on the downside than on the upside. Also price volatility has a positive effect on the 'accelerated' depletion rate of the mine. This can be seen from the first term of the PDE in equation (4-14). Within most part of the price sensitive region, the 'accelerate' component of virtual depletion tends to dominate the 'restock' component. For constant parameters, increase in price volatility drives the 'accelerate' component to be more negative since in this case reserve has a diminishing marginal sensitivity to price (Rss<0). Given that 2DPM and CRM generally have opposing response to volatility, the degree that CRM values an asset more than 2DPM is higher with increase in volatility of the underlying commodity price.

Figure 4-14 shows the value of the mine for different development lags. As expected, the longer the development lag the lower the value of the mine. At a copper price of 1.5$/lb the value of the mine is shown as $932.28million if it can be developed in two years, $874.37million if developed in four years and

$819.85million in six years. This loss in value is principally due to delayed cash-flow from operation if development takes long to complete. At very low commodity prices, however, it is seen that longer development lag adds value as shown in Figure 4-15. This value is primarily derived from the flexibility to close or abandon the mine at lower prices. Also longer development period increases the chance that prices will be favourable after development.

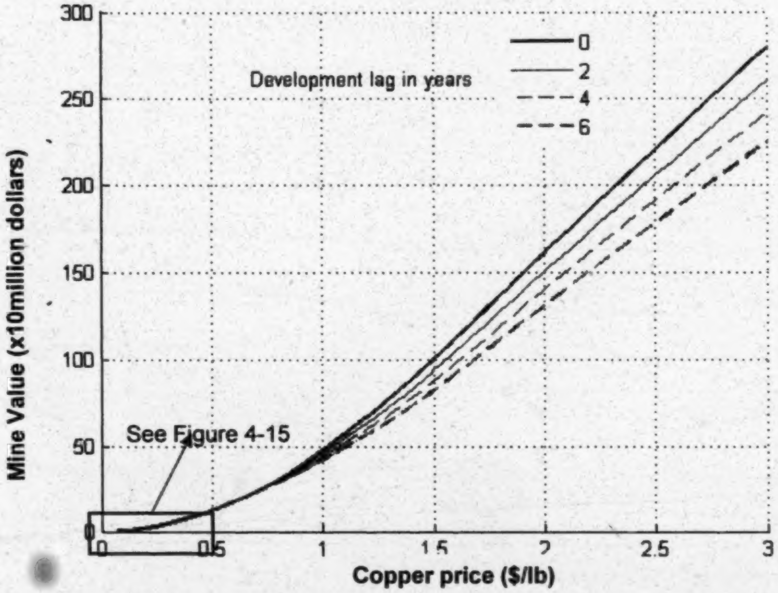

Figure 4-14: Value of a Developed Mine for Different Development Lag

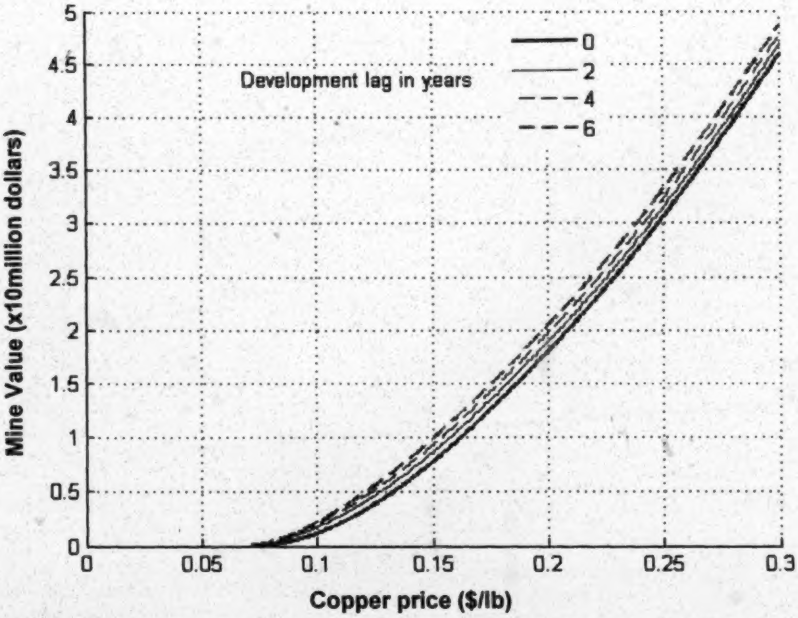

Figure 4-15: Value of a Developed Mine for Different Development Lag at Low Commodity Price.

4.7 Summary and Conclusions

Price uncertainty and reserve variability are the major source of risks faced by a resource firm. Current continuous time valuation models however treat reserve as known and constant which is a significant shortcoming. To address this shortcoming the equilibrium partial differential equations that govern the value of a mine have been developed by simultaneously considering both price uncertainty and reserve variability. The result is a one factor model with cut-off grade flexibility. Reserve is modeled as known but subject to changes due to cut-off grade flexibility at the disposal of management in response to price swings. The change in cut-off grade in response to price changes is however exogenous to the model. Under the 2-D price model (2DPM), the reserve process can be characterised by a Brownian motion with a drift term η which could either be less or greater than the drift (physical depletion, q) in the constant reserve model (CRM). The stochastic component is proportional to the sensitivity of the reserve-price curve and is completely defined by the Weiner process that characterises the commodity price.

The results from 2DPM, compared with that from CRM, indicate that the constant reserve model generally predicts higher asset values than the corresponding 2DPM model with the same level of reserve as potential reserve. This difference influences the investment and abandonment decisions. However, this may not always be the case when the result from 2DPM is compared with that from CRM with a lower level of reserve (but higher quality reserve). This shows that the ad-hoc flexible cut-off grade management presented here may not necessarily add value to a mine. Cut-off grade determination should therefore not be made in isolation without consideration of the operating policy[45] that maximises mine value. This is especially so when one is dealing with vein type deposits or deposits with higher grade variability. On the other hand, deposits with low grade variability may be seriously undervalued if the potential reserve is not taken into account during valuation since for such deposits quantity of ore has more dominant effect on value than quality.

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[45] Operating policy here refers to the threshold prices at which an opened mine is closed, a closed mine is reopened, a mine is abandoned and the threshold price at which the investment decision is made.

The exercised boundaries at which a closed mine is re-opened and an opened mine is closed depends on the relative values of the open and close options. For the model parameters, 2DPM predicts that operating mines will close earlier and closed mines will re-open earlier than mines under CRM. Lower project values from 2DPM require that mineral assets be abandoned earlier than assets that are valued using CRM. Also lower 2DPM values make the critical price at which firms commit to development and investment to be higher than that predicted by CRM. Under ad-hoc flexible cut-off grade policy such as the one prescribed in this model, firms may wait longer than necessary to invest in mineral opportunities especially if the deposit has higher variability in grade. Although longer waiting times mean less investment under the 2DPM, firms have the opportunity to invest under more favourable conditions and be subject to less risk due to price swings.

5 2D CTSP PRICE-RESERVE MODEL – (2DPR)

5.1 Introduction and Modeling Philosophy

Besides price changes that could lead to the change in reserve base, reserve could also change as a result of the estimation variance of the ore grade.[46] In this case the stochastic process that characterizes the reserve base is governed by the sum of two processes. By describing the stochastic processes that govern the evolution of price and grade over time it is possible to completely characterize the stochastic process that governs the reserve. The price process is assumed to be governed by a one factor Weiner process as given in equation (3-1).

5.2 Reserve Process

The stochastic process underlying grade is also assumed to be a Weiner process with no drift component as given in equation (5-1)[47]. The Weiner process is appropriate in describing the grade process since the current grade distribution takes into account all past drilling and sampling information. In that sense, the future distribution of grade depends on only current grade distribution alone. This is consistent with the properties of the Weiner process. Equation (5-1) also means that grades are lognormally distributed and that the relative change in grade is a normally distributed random variable with an expected value of zero and a variance proportional to time lag between evaluations. Equation (5-2) states that the reserve depends on the price of the commodity, the average grade and the state of the mine. Using Ito's Lemma to expand equation (5-2), the instantaneous change in reserve can be formulated by equation (5-3), which states that the dynamics governing the reserve process is determined by the production rate and the stochastic processes underlying both the grade and price dynamics.

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[46] It is important to clarify that this change in reserve is not due to new discovery as a result of exploration activities but is only as result of the variance associated with the ore grade estimation and the increase in knowledge as more information is gathered about ore grades.

[47] The use of the Weiner process to characterize the stochastic process underlying the ore grade suggests that there is no "abnormal" information revealed during the mining phase to cause a sudden "jump" in reserves. This is particularly true for low grade massive deposits like copper. Alternatively one could look at a combination of a jump and a diffusion process that could account for the possibility of abrupt changes in reserves due to some extra-ordinary information arising from grade control during mining.

$$dG = \sigma_G G dw \qquad (5\text{-}1)$$

$$R = (S, G; j) \qquad (5\text{-}2)$$

$$dR = -qdt + R_s dS + \frac{1}{2} R_{SS} (dS)^2 + R_G dG + \frac{1}{2} R_{GG} (dG)^2 \qquad (5\text{-}3)$$

Equation (5-4) is the result by substituting equations (3-1) and (5-1) into equation (5-3).

$$dR = (-q + (\alpha - \partial)\, SR_S + \frac{1}{2} R_{SS} \sigma^2 S^2 + \frac{1}{2} R_{GG} G^2 \sigma_G^2) dt + R_s \sigma S dz + R_G \sigma_G G dw \qquad (5\text{-}4)$$

Equation (5-4) gives the stochastic process that describes the changes in reserve. σ_{RG} is the volatility in change in reserve due to changes in grade alone and σ_{RS} is the volatility of change in reserve due to price changes alone. σ_{RG} will normally reduce in time as the sampling rate is increased and more knowledge is gained about ore grades during the production phase of the project. The change in reserve has effective volatility σ_{RE} given by equation (5-8) which is greater than the volatility when the changes are due to only prices as in the 2DPM. It is assumed that there is no correlation between the Weiner processes underlying the price and grade processes.

$$\eta_1 = (\alpha - \partial)\, SR_S + \frac{1}{2} R_{SS} \sigma^2 S^2 + \frac{1}{2} R_{GG} G^2 \sigma_G^2 \qquad (5\text{-}5)$$

$$\sigma_{RS} = R_S \sigma S \qquad (5\text{-}6)$$

$$\sigma_{RG} = R_G \sigma_G G \qquad (5\text{-}7)$$

$$\sigma_{RE}^2 = R_S^2 \sigma^2 S^2 + R_G^2 \sigma_G^2 + 2\rho_{SG} \sigma_{RG} \sigma_{RS} \qquad (5\text{-}8)$$

$$\sigma_{RE}^2 = R_S^2 \sigma^2 S^2 + R_G^2 \sigma_G^2 G^2 \qquad (5\text{-}9)$$

Setting ρ_{SG} to zero reduces equation (5-8) to equation (5-9). The drift term in this case is also greater than that of the 2D price model since typically R_{GG} is greater than zero.

Equation (5-4) can be written as equation (5-10) by substituting into it equations (5-5), (5-6) and (5-7) . Equation (5-10) describes the change in reserve as a stochastic variable with expected value η_1 and volatility σ_{RE}.

$$dR = (-q + \eta_1)\, dt + \sigma_{RS} dz + \sigma_{RG} dw \qquad (5\text{-}10)$$

The form of the reserve process in equation (5-4) is difficult to simulate. Since the level of reserve is uncertain, the grade-tonnage curve is uncertain and so are the reserve-price and reserve-grade sensitivity curves. Equation (5-4) thus describes a reserve process with both stochastic drift and volatility terms. Using this form of the reserve process therefore introduces five other stochastic 'variables' which makes the solution non tractable. There is however a tradeoff between making "simplifying" assumptions in order to enhance the solveability of the resulting model and keeping the true interaction of the underlying variables which will result in a complex model with little possibility of solving.

The SR_S, S^2R_{SS}, GR_G and G^2R_{GG} terms in equation (5-4) can be written as some functions of R so the terms multiplying dt in equation (5-4) can be written as equation (5-11).

$$\psi(R) = (\alpha - \partial)\, SR_S + \frac{1}{2}R_{SS}\sigma^2S^2 + \frac{1}{2}R_{GG}G^2\sigma_G^2) \qquad (5\text{-}11)$$

Also the sum of two Weiner processes is a Weiner process so that the last two terms of equation (5-4) can be written as equation (5-12). The reserve process can be rewritten in the form shown in equation (5-13), which is a generalized Ito process.

$$\xi(R)\overline{dw} = R_s\sigma Sdz + R_G\sigma_G Gdw \qquad (5\text{-}12)$$

$$dR = (-q + \psi(R))dt + \xi(R)\,\overline{dw} \qquad (5\text{-}13)$$

If the net relative change in reserve (changes not due to periodic extraction) is assumed to be normally distributed with a constant mean and standard deviation then equation (5-14) is a special case of equation (5-13).

$$dR = (-q + \phi R)dt + \bar{\sigma} R\, \overline{dw} \qquad (5\text{-}14)$$

Equations (5-14) and (5-4) are similar but equation (5-14) does not require the knowledge of the reserve-price and reserve-grade sensitivity curves, which are also stochastic. ϕ in equation (5-14) captures the expected annual growth rate in reserve, which could be from changes in price or grades or both, in which case it could be positive or negative. It could additionally capture growth in reserve due to exploration activities. $\bar{\sigma}$ is the volatility or standard deviation in the relative change of reserve. In this model ϕ and $\bar{\sigma}$ are assumed to be constant although in reality they will change with time. In particular the uncertainty in reserve, $\bar{\sigma}$, will decrease with time as more information is gathered regarding ore grade distribution from additional drilling during mining.

5.3 Generalized Mine Value Model

Given the reserve process in equation (5-14), the price process in equation (3-1) and the general mine model in equation (3-19), the instantaneous change in the market value of the mine can be written as in equation (5-15).

$$\begin{aligned} dM = & \left[(-q + \phi R)M_R + S(\alpha - \partial)M_S + \frac{1}{2}\overline{\sigma^2} R^2 M_{RR} + \frac{1}{2}\sigma^2 S^2 M_{SS} + M_t \right] dt \\ & + \bar{\sigma}\, R M_R\, \overline{dw} \quad + \sigma S M_S\, dz \end{aligned} \qquad (5\text{-}15)$$

The derivation of the equilibrium partial differential equation satisfied by the mine is in the framework of the equilibrium model proposed by Constantinides (1978)[48]. Constantinides (1978) showed that, under the assumption of free entry by firms, no firm realizes windfall profits by undertaking a project. Therefore assuming that the CAPM holds, the equilibrium expected return on a project α_P must satisfy equation (5-16).[49]

$$\alpha_P - r = \lambda \sigma_{Pm} / \sigma_m \tag{5-16}$$

$$\lambda = (\alpha_m - r) / \sigma_m \tag{5-17}$$

λ is the market price per unit of risk defined as in equation (5-17). The subscript *m* refers to the market portfolio and σ_{PM} is the covariance between the project returns and the returns on the market portfolio.

In this context, the expected growth rate in metal price α from equation (3-1) can be written as equation (5-18).

$$\alpha - r = \lambda \sigma_{Zm} / \sigma_m \tag{5-18}$$

$$\sigma_{Zm} = \sigma \sigma_m \rho_{Zm} \tag{5-19}$$

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[48] "Reserve" or a derivative of "reserve" is not traded in the capital market. The approach by Constantinides is used to derive the equilibrium condition instead of portfolio replication since portfolio replication requires the existence of other traded portfolio of assets or derivatives on these assets that can be used to hedge the risk of the underlying.

[49] This equation is the intertemporal capital asset pricing model by Merton (1973).

σ_{ZM} is the covariance between the Weiner process dz and the market returns. Substituting equation (5-19), which is the definition for covariance, in equation (5-18) gives equation (5-20).

$$\alpha - \mathrm{r} = \lambda \rho_{ZM} \sigma \tag{5-20}$$

ρ_{ZM} in equation (5-20) is the coefficient of correlation between the Weiner process dz and the market returns.

In the same way, the expected return from undertaking a mining project should be consistent with equation (5-16). The instantaneous return on holding a long position in the mine, r_p, is given by the sum of any capital appreciation or depreciation in the market value, *dM*, and the cash return (*Adt*) generated by the project as shown in equation (5-21).

$$r_p = \frac{dM + Adt}{M};\ A(S,q) \tag{5-21}$$

$$\begin{aligned} r_p = {} & \frac{1}{M}\left[(-q+\phi R)M_R + (\alpha-\delta)SM_S + \frac{1}{2}\overline{\sigma}^2 R^2 M_{RR} + \frac{1}{2}\sigma^2 S^2 M_{SS} + \frac{(M_t + A)}{M}\right]dt \\ & + \frac{\overline{\sigma} R M_R \overline{dw} + \sigma S M_S dz}{M} \end{aligned} \tag{5-22}$$

Substituting equation (5-15) into equation (5-21) gives equation (5-22). Equation (5-22) states that, the total return on the mine follows a stochastic process and has two components; a drift or expected component (the first part of the equation (5-22) multiplying dt) and a random part. This is analogous to equation (3-1) of the price process. The expected return on the mine is thus given by equation

(5-23). Equation (5-24) gives the covariance between the return on the mine and the market returns.[50]

$$\alpha_p = \frac{1}{M}\left[(-q+\phi R)M_R + (\alpha - \delta)SM_S + \frac{1}{2}\overline{\sigma}^2 R^2 M_{RR} + \frac{1}{2}\sigma^2 S^2 M_{SS} + M_t + A \right] \tag{5-23}$$

$$\sigma_{PM} = \frac{\sigma_M}{M}\left[\rho_{Zm}\sigma SM_S + \rho_{\overline{W}m}\overline{\sigma}RM_R \right] \tag{5-24}$$

ρ_{Zm} and ρ_{Wm} are the instantaneous correlation between the market return and the independent Weiner processes dz and $\overline{dw}$ respectively. Equation (5-25)[51] is the result of substituting equations (5-23) and (5-24) into equation (5-16).

$$\begin{aligned} &\left(-q+(\phi - \lambda\rho_{\overline{w}m}\overline{\sigma})R\right)M_R + S(r-\delta)M_S + \frac{1}{2}\left(\overline{\sigma}^2 R^2 M_{RR} + \sigma^2 S^2 M_{SS}\right) \\ &-rM - M_\tau + A = 0 \end{aligned} \tag{5-25}$$

$$\left(-q+(\phi - \lambda_R)R\right)M_R + S(r-\delta)M_S + \frac{1}{2}\left(\overline{\sigma}^2 R^2 M_{RR} + \sigma^2 S^2 M_{SS}\right) - rM - M_\tau + A = 0 \tag{5-26}$$

$$\lambda_R = \lambda\rho_{\overline{w}m}\overline{\sigma} \tag{5-27}$$

Equation (5-26) is the equilibrium partial differential equation that must be satisfied by the value of a mine subject to the appropriate terminal and boundary conditions. λ_R in equation (5-26) is the market price of reserve risk it is defined in equation (5-27). The equilibrium conditions for different states of the mine can be formulated with the appropriate terminal and boundary conditions. The coefficients of the second order terms in equation (5-26) show that if both price and reserve volatilities are non zero, then the governing equation is an elliptic differential equation for all values of price and reserve greater than zero.[52]

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[50] Note that Covariance(x+y, m)=Covariance(x,m) + Covariance(y,m).

[51] It is important to note that $\alpha-\lambda\rho_{Zm}\sigma = r$ as shown in equation (5-20)

5.3.1 Dynamic Value of an Operating Mine

The inflation adjusted value of an operating mine O(S, R, t; o) is given by equation (5-28) subject to equations (5-29) to (5-32):

$$\left(-q+(\phi-\lambda_R)R\right)O_R+S(r-\delta)O_S+\frac{1}{2}\left(\overline{\sigma}^2R^2O_{RR}+\sigma^2S^2O_{SS}\right)-rO+A=0 \qquad (5\text{-}28)$$

$$O(0,R;o)=0 \qquad (5\text{-}29)$$

$$O(S,0;o)=0 \qquad (5\text{-}30)$$

$$O_{RR}(S,R;o)=0;R\to\infty \qquad (5\text{-}31)$$

$$O_{SS}(S,R;o)=0\ ;\ S\to\infty \qquad (5\text{-}32)$$

Equations (5-29) and (5-30) are the zero reserve and zero price conditions respectively. Equation (5-31) sates that, at extreme reserve values the value of the mine is linear in reserve. Equation (5-32) states that when the price becomes very large the mine value is linear in S, consistent with linear cash-flow model (Laughton and Jacoby, 1991).

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[52] $-4\sigma^2\overline{\sigma}^2S^2R^2<0\ ;\ \forall\,\sigma\neq 0$ and $\overline{\sigma}\neq 0$

When an operating mine has the flexibility to temporarily shut down or abandon in the event of economic downturns, the zero reserve and zero price conditions must be replaced by equations (5-33) to (5-35).

$$O(S_c^*,R_c^*;o) = \max(C(S_c^*,R_c^*;c) - Kc, 0) \qquad (5\text{-}33)$$

$$O_S(S_c^*,R_c^*;o) = C_S(S_c^*,R_c^*;c) \qquad (5\text{-}34)$$

$$O_R(S_c^*,R_c^*;o) = C_R(S_c^*,R_c^*;c) \qquad (5\text{-}35)$$

These are the value-matching and smooth pasting conditions that ensure that the switch from open to close is optimally done. Equation (5-33) states that, at any time during the operating stage, the closure and abandonment options should be considered in determining the value of the mine. One has to evaluate if the mine should continue to operate or it is optimal to exercise the close option and receive the value of a closed mine or exercise the abandon option and incur the cost of abandonment.

For reserve and price uncertainty one has to determine reserve and price region for which it is optimal to abandon a closed mine, close an open mine, re-open a closed mine or invest in the project. In this case the Sc* and Rc* are both vectors.

5.3.2 Dynamic Value of a Closed Mine

For a closed mine the production rate is zero, and thus no positive cash-flows from operation. There is however a periodic maintenance cost Ac, incurred to ensure that re-opening is possible. The equilibrium equation for the value of the closed mine C(S, R; c) is given by equation (5-36) subject to equations (5-37) to (5-41).

$$(\phi-\lambda_R)RC_R+S(r-\delta)C_S+\frac{1}{2}\left(\overline{\sigma}^2R^2C_{RR}+\sigma^2S^2C_{SS}\right)-rC+Ac=0 \qquad (5\text{-}36)$$

$$C(S,0;c)=0 \qquad (5\text{-}37)$$

$$C(0,R;c)=0 \qquad (5\text{-}38)$$

$$C(S_o^*,R_o^*;c)=O(S_o^*,R_o^*;o)-Kco \qquad (5\text{-}39)$$

$$O_S(S_o^*,R_o^*;o)=\quad C_S(S_o^*,R_o^*;c) \qquad (5\text{-}40)$$

$$O_R(S_o^*,R_o^*;o)=\quad C_R(S_o^*,R_o^*;c) \qquad (5\text{-}41)$$

Equations (5-37) and (5-38) are the zero reserve and zero price conditions. Equation (5-39) states that, at a critical price and reserve a closed mine is re-opened to receive an open mine by incurring a re-opening cost, *Kco*. Equations (5-40) and (5-41) are the two smooth pasting conditions which ensures optimal switch from close to re-open.

$$C(Sa^{*},Ra^{*};c)=0 \tag{5-42}$$

$$C_S(Sa^{*},Ra^{*};c)=0 \tag{5-43}$$

$$C_R(Sa^{*},Ra^{*};c)=0 \tag{5-44}$$

Equations (5-37) and (5-38) are replaced by equations (5-42) to (5-44) if there is the option to abandon. Equation (5-42) means that the mine is abandoned at a net cost of zero at certain critical reserve and price and equations (5-43) and (5-44) ensures optimal abandonment.

5.3.3 Dynamic Value at Mine Development Stage

Similarly, the value of the mine at the stage of development, assuming that all development expense is at the beginning of development, satisfies equation (5-45) subject to equations (5-46) to (5-50).

$$(\phi-\lambda_R)RD_R+S(r-\delta)D_S+\frac{1}{2}\left(\bar{\sigma}^2R^2D_{RR}+\sigma^2S^2D_{SS}\right)-rD-D_\tau=0 \tag{5-45}$$

$$D(S,R,T_d;d)=\max(O(S,R;o),C(S,R;c)) \tag{5-46}$$

$$D(S,0,\tau_d;d)=0 \tag{5-47}$$

$$D(0,R,\tau_d;d)=0 \tag{5-48}$$

$$D_{RR}(S,R,\tau_d;d)=0;R\to\infty \tag{5-49}$$

$$D_{SS}(S,R,\tau_d;d)=0\ ;\quad S\rightarrow\infty \tag{5-50}$$

Equation (5-46) implies that, the value at the end of development is equal to the value of the maximum of opened mine or closed mine. As noted earlier this will always be equal to the value of an opened mine if the decision to develop is optimal. Equations (5-47) and (5-48) are the zero reserve and price conditions respectively and equations (5-49) and (5-50) are respectively the maximum reserve and price conditions.

5.3.4 Dynamic Value at the Investment Decision Stage

The investment decision constitutes the choice to exercise the option to invest by paying for the cost of development and in return receive an opened mine with all of its embedded options. The value of the mine, at this stage, satisfies equation (5-51) subject to equations (5-52) to (5-57).

$$(\phi-\lambda_R)RV_R+S(r-\delta)V_S+\frac{1}{2}\left(\bar{\sigma}^2R^2V_{RR}+\sigma^2S^2V_{SS}\right)-rV-V_\tau=0 \tag{5-51}$$

$$V(S,R,T_u;u)=\max(D(S,R,\tau_d;d)-Io,0) \tag{5-52}$$

$$V(S,0,\tau_u\ ;u)=0 \tag{5-53}$$

$$V(0,R,\tau_u\ ;u)=0 \tag{5-54}$$

Equation (5-52) states that when the lease period to initiate development is exhausted, the value of the undeveloped mine depends on whether it is optimal to pay the development cost and receive a developed mine or give the concession up. Equations (5-53) and (5-54) are the zero reserve and zero metal price conditions.

$$V(Si^*,Ri^*,\tau_u;u)=D(Si^*,Ri^*,\tau_d\,;d)-Io \qquad (5\text{-}55)$$

$$V_S(Si^*,Ri^*,\tau_u;u)=D_S(Si^*,Ri^*,\tau_d\,;d) \qquad (5\text{-}56)$$

$$V_R(Si^*,Ri^*;u)=D_R(Si^*,Ri^*,\tau_d\,;d) \qquad (5\text{-}57)$$

Equation (5-55) is the value-matching condition on optimal exercise of the investment option. Equation (5-56) means that at any time during the lease period, if price hit a certain critical price and reserve is at a certain critical level the firm pays for the development cost to receive a developed mine. Equations (5-56) and (5-57) are the two smooth pasting conditions required to ensure an optimal switch between wait and invest.

5.4 Case Study of the 2D CTSP Price-Reserve Model

The example below is used to illustrate the nature of the solution provided by this model. The same economic data in

Table 4-1 is used to facilitate comparison of the solutions from CRM and 2DPR. The market price of reserve risk is assumed to be zero since the correlation between the Weiner process that characterises reserve $\overline{dw}$ and the market return is zero. The expected growth rate in reserve is assumed to be zero since this assumption fairly holds for a matured and well explored property[53]. Reserve volatility is assumed to be 15%. A reserve volatility of zero will violate the model assumption that requires reserve volatility to be nonzero[54] and this will reduce the model to a corresponding one factor model of CRM. There has been some work to empirically determine these parameters. Slade (2001) used published annual reserve data from copper mining companies in British Colombia and Quebec between 1980 and 1993 and found parameter values to be up to 4% for the expected growth rate (ϕ) and standard deviation of this estimate (volatility) to be 15%. These parameters are therefore location or even mine specific and may not be used as generic numbers. The purpose of this model is however to demonstrate that in the presence of reserve uncertainty, which is a norm in the mineral industry, a constant reserve assumption in valuation of assets may not give the true market value of mineral assets.

The model equations described in the sections above are solved using FEMLAB and MATLAB as described in Section 4.5[55]. Development is again assumed to be instantaneous and the option to invest is assumed to be perpetual[56]. It is also assumed that the mine can be abandoned at any time at a net cost of zero to the mine operator.

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[53] It must however be emphasized that firms continue to carry out exploration activities even after the investment decision in order to better understand the characteristics of the ore-body.

[54] See footnote 52

[55] See Section B6 in Appendix B for further implementation details

[56] See footnote 33.

For the purpose of carrying out a sensitivity analysis on model parameters ϕ and reserve volatility ($\bar{\sigma}$), the mine is assumed to operate with a fixed operating policy. Under this policy, it is assumed that the mine can shut down at no cost if price is less than the marginal operating cost and also re-open at no cost if price is greater than marginal cost. Although these assumptions make the solution process simple in the sense that one need not solve coupled PDEs and determine the exercise boundaries, there is no loss of generality. The close, as well as, the reopen price boundaries in this case are both pre-determined as the marginal cost since there is no shutdown and reopen costs. The abandon price is also predetermined as zero since there is no maintenance cost during the period of closure.

Table 5-1: Input Data to Model 2DPR

Average grade; AG	0.65%
Cost per output of copper, C	\$0.65/lb
Extraction rate, q	5mt/year
Real risk-free interest rate; r	6%
Convenience yield δ	4%
Output price volatility σ	20%
Volatility in reserve $\bar{\sigma}$	15%
Expected growth rate of reserve ϕ	0
Market price of reserve risk , λ_R	0
Maintenance cost, Ac	\$0.5m/year
Closure Cost, Kc	\$20m
Re-Open Cost , Kco	\$20m
Investment Cost, Io	\$150m

5.5 Discussion of Results

For the purpose of comparison with the constant reserve model the resulting solution is evaluated at a reserve level of 127.8mt. Figure 5-1 shows the results from 2DPR compared with the results from CRM and NPV.

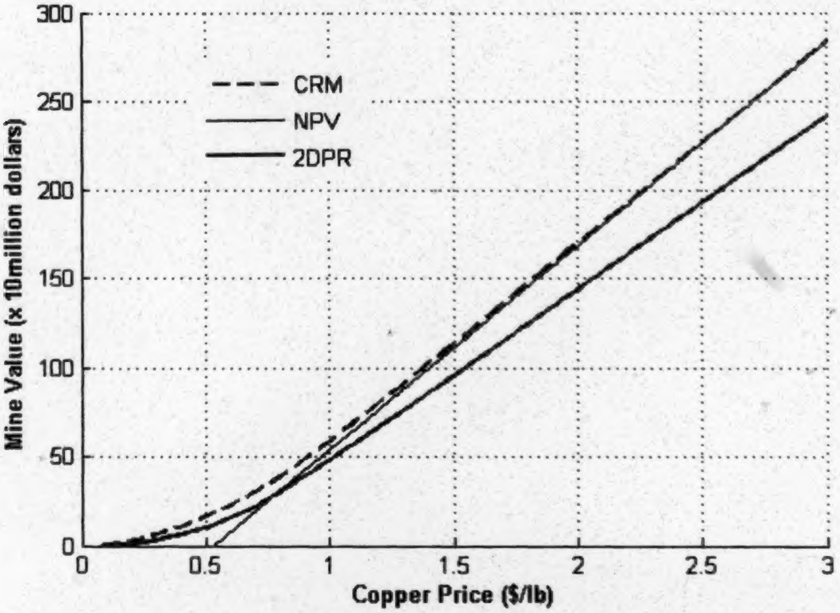

Figure 5-1: Mine Value using 2DPR vs. CRM and Static NPV.

The value from 2DPR compared with that from NPV analysis shows that generally within the region where there is the flexibility to exercise the close and abandon options 2DPR has higher values than the equivalent NPV model. Beyond this region, the constant reserve assumption used in NPV analysis makes the value from NPV higher. At a price of $0.7/lb, the value from 2DPR is $218 million compared with $194 million from NPV and at a price of $1.0/lb

2DPR has a value of $480.7 million compared with $538.2 million from NPV. Figure 5-1 also shows that the value of the mine under 2DPR is lower than that from the corresponding CRM. At copper price of $1.5/lb the value of the mine using 2DPR is $952 million which is about $178 million (16%) less than the value from CRM.

The introduction of reserve uncertainty in addition to price uncertainty in principle increases the overall risk of the project. In general, the literature on options suggests that increase in uncertainty in the underlying should increase the value of flexibility and hence the value of the option on the asset (Merton, 1973; Black and Scholes, 1973; Smith, 1976; Dixit and Pindyck, 1994; Huchzermer and Cohen, 1999). This increase in value is basically due to the asymmetric nature of the option value. The results presented here may therefore seem counter intuitive. These results are however to be expected since for a fully explored property, if the expected growth rate in reserve is zero any uncertainty in this expectation makes the property less attractive than the corresponding 'certainty equivalent' property. The lower value can also be explained by the fact that reserve risk is private[57] to the firm that owns the property and the options on the property. Since this risk is private, the market does not pay a premium for it and so any market based valuation approach should assign a lower value to the asset as well as the option on the asset.

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[57] Reserve risk is private to the firm (even publicly traded firms) in the sense that unlike risks due to certain macroeconomic variables (like price, demand and inflation), the changes in reserve of a particular firm does not influence other firms in the market (or industry as a whole).

Vollert (2003) and Huchzermeier and Cohen (1999) established that private non market risks generally decrease the option value, as well as, the value of the firm. The model presented here is consistent with this assertion as can be seen in Figure 5-2. Figure 5-2 examines the sensitivity of the mine value to changes in reserve volatility. The mine value decreases with increasing reserve volatility. Thus the degree to which the value of a mine from the constant reserve assumption deviates from the 'true' value depends on the uncertainty in reserve.

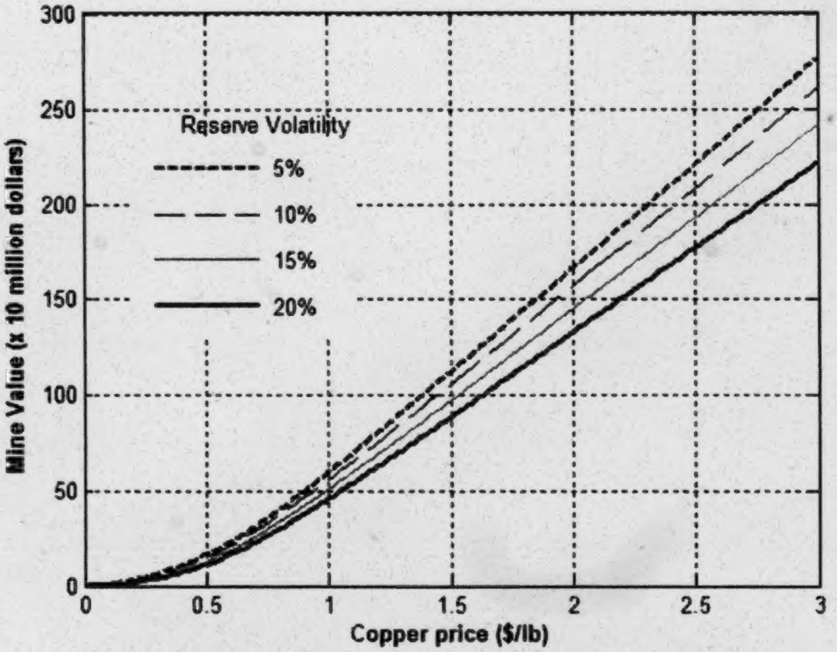

Figure 5-2: Mine Value for Different Reserve Volatilities

Mine value determined using 2DPR is always lower than that from CRM as long as the expected growth rate in reserve is zero. Figure 5-3 shows the value of the mine for different expected growth rate in reserve. Compared with the value from

CRM (for the given parameter values) if the expected growth rate is greater than 2% the value from CRM is generally lower than that from 2DPR.

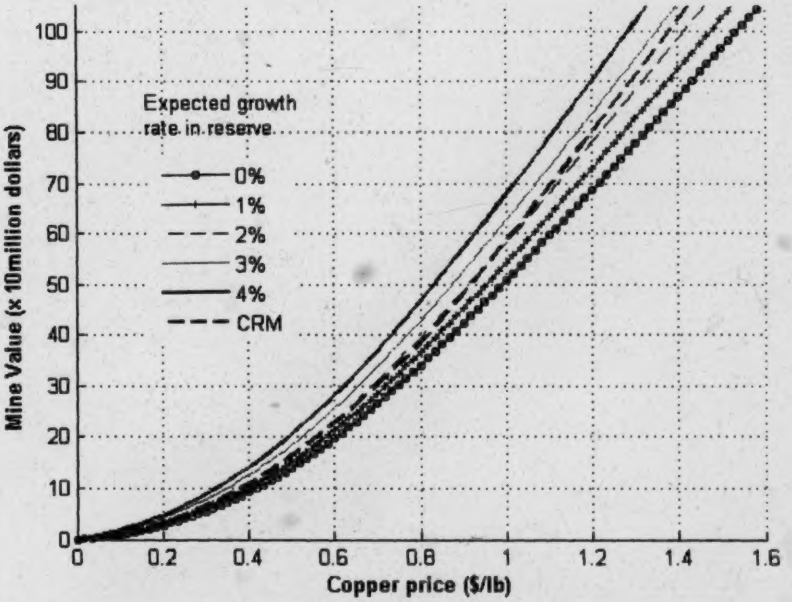

Figure 5-3: Mine Value for Different Reserve Growth Rates

Figure 5-3 illustrates that for a "green-field" mineral property[58], using a constant reserve assumption may under-value the property. The problem that one faces however is the estimation of the expected growth rate in reserve. It is however an established fact that, virgin mineral properties do not sell only for their proven and probable reserve but also receive a premium that depends on the level of other mineral resource (Lawrence (2000); Semeniuk (2001)). In this sense, the ratio of indicated resource to the sum of proven and probable reserve, as well as, expected annual exploration expense can serve as a guide in estimating ϕ.

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[58] where there is the expectation of additional reserves during mining operations by the conversion of indicated resources to proven or probable reserves

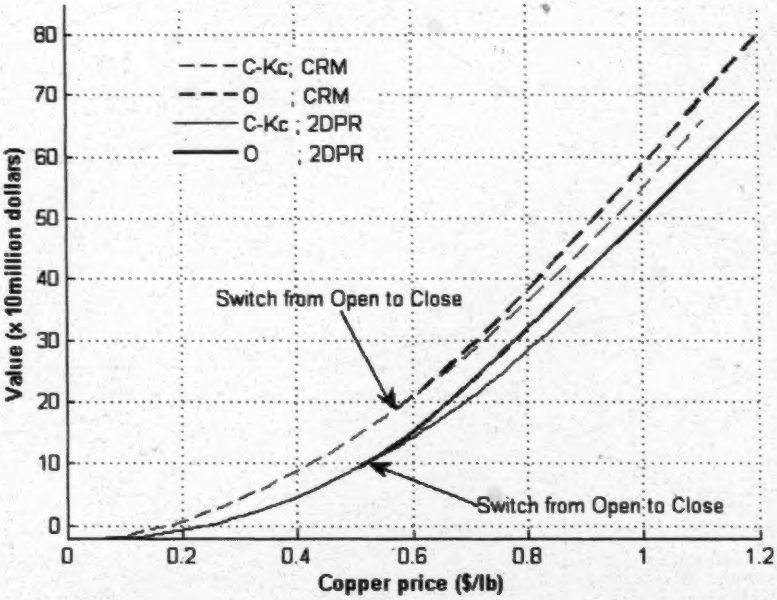

Figure 5-4: Open and Closed Values under 2DPR compared with CRM

Figure 5-4 shows open and close values, as well as, the threshold price to switch from open to close and Figure 5-5 shows the threshold price to switch from close to open for both CRM and 2DPR models. The close threshold ($0.49/lb) is lower in 2DPR than that in CRM ($0.56/lb). This means that an operating mine under 2DPR is more attractive relative to a closed mine and so it operates for a longer time even at depressed commodity prices. In the same way a closed mine gets re-opened earlier in 2DPR ($0.92/lb) than CRM ($1.10/lb).

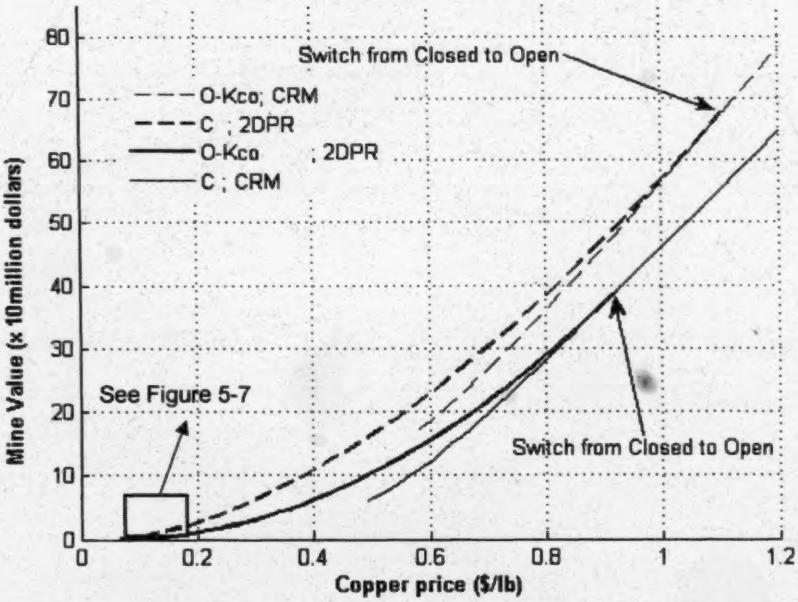

Figure 5-5: Open and Closed Values under 2DPR compared with CRM

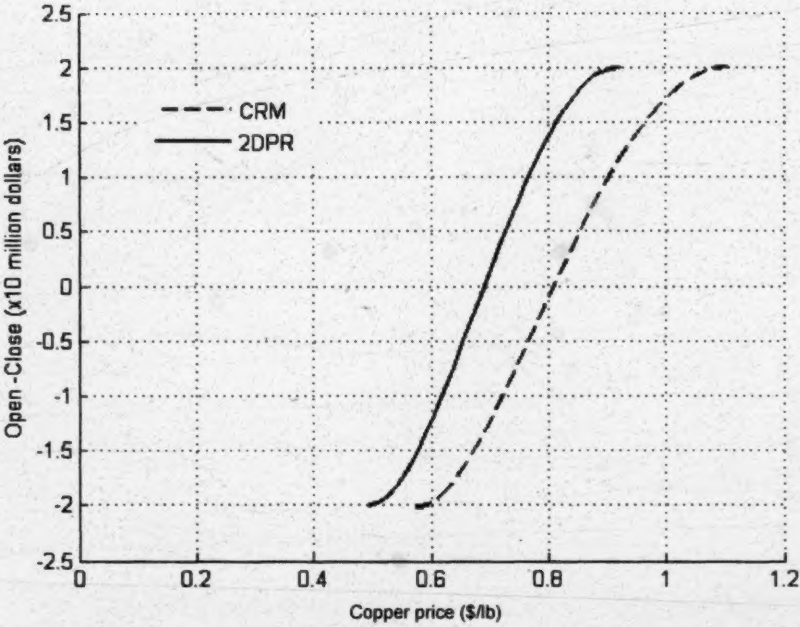

Figure 5-6: Difference between Open and Close in the Inactive Region

Figure 5-6 shows the difference between open and close within the price inactivity region[59]. The difference for 2DPR is greater everywhere compared with CRM. This means that an opened mine is more attractive than a closed mine under 2DPR than it is for CRM. A closed mine however is abandoned earlier ($0.082/lb) when there is reserve uncertainty than the case where reserve is constant ($0.056/lb) as shown in Figure 5-7.

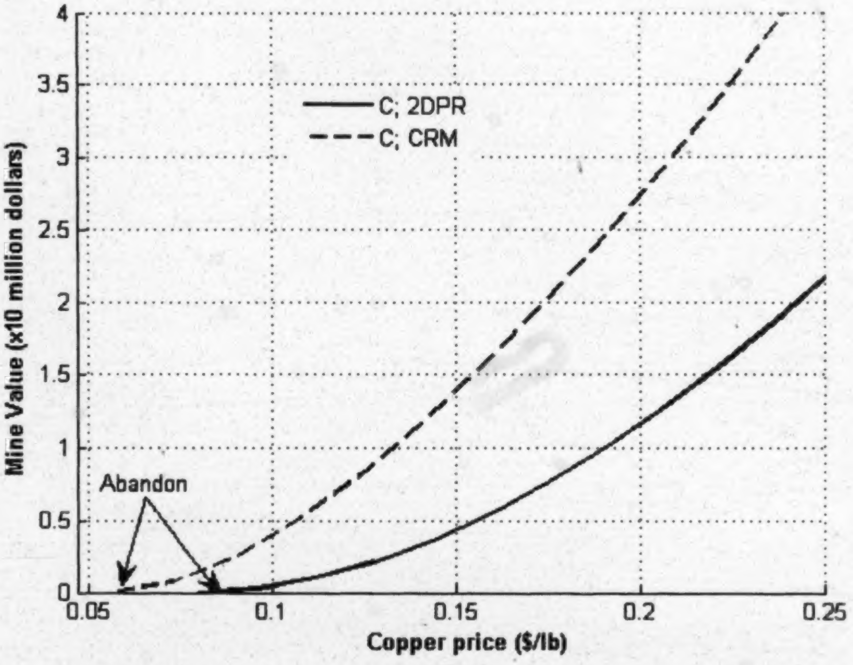

Figure 5-7: Value of Closed mine showing abandonment threshold

Early abandonment can be explained by the fact that there is no further information revealed about reserves when a mine is closed so that the switch from close to abandon is highly influenced by maintenance cost. Since a closed mine under CRM has higher value than that from 2DPR, at the same maintenance cost, it is expected that a 2DPR mine will be abandoned earlier.

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[59] In this region a closed mine remains closed and an opened mine remains opened. This is the interval between the close and re-open threshold prices.

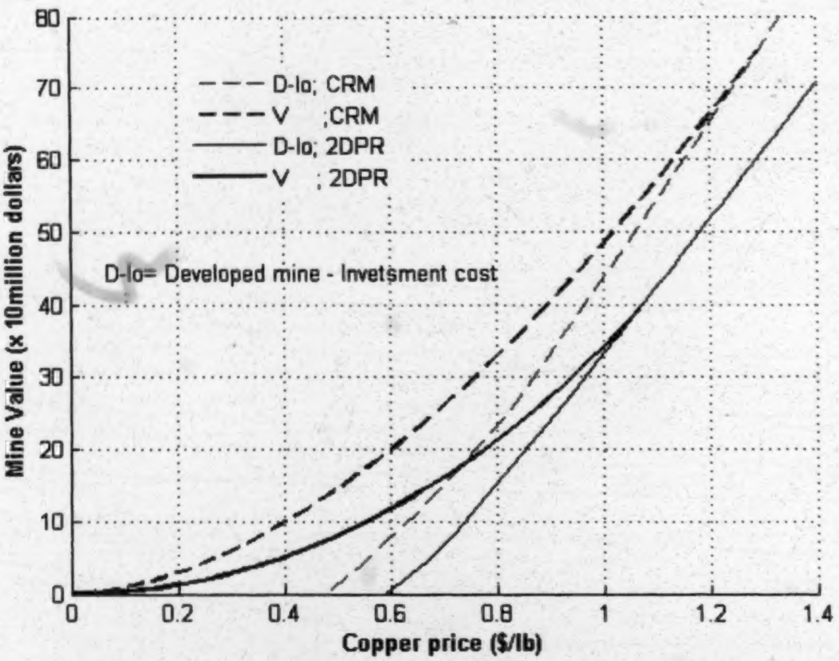

Figure 5-8: Value of the Option to invest; 2DPR compared with CRM

Figure 5-8 shows the value of the option to invest and the threshold investment price for both CRM and 2DPR models. It is interesting to find out how reserve uncertainty affects the decision to invest through the investment threshold price. At the same reserve level, the 2DPR mine has a lower value than that from CRM, but at the same capital requirement 2DPR predicts earlier investment than CRM. The investment threshold under 2DPR is $1.07/lb whilst that under CRM is $1.29/lb. This result might seem counter intuitive. Here the decision to invest is jointly determined by the optimal level of both reserve and commodity price. At an optimal reserve level of 127.8mt, although the mine value is lower than the value of the corresponding mine using CRM; 2DPR model suggests that this reserve level is high enough to warrant early investment even at relatively lower prices.

Increase in volatility or uncertainty in reserve decreases the value of the firm's investment option (the opportunity cost of investing) even more than the value of the mine as shown in Figure 5-8. In this case, the value in waiting which is the difference between the option value and the net investment value also diminishes quickly. This result is consistent with Vollert (2003) that suggests that private (non market) risk that is not assigned a premium in capital markets decreases the firms option to invest and also decreases the threshold value at which the firm invests. In the study, he defined competitive entry risk as exogenous and private and concluded that the decision to invest earlier when this risk exists is possibly to have the early mover advantage and earn potential monopoly profits.

In the case of reserve risk however, a possible explanation will be how uncertainty is resolved through time. For a deposit, the reserve uncertainty is not necessarily resolved through waiting on the decision to invest; the actual level of reserve is never known until exploitation has begun. Exploitation requires investment but this model does not consider stage development where the level of investment depends on information revealed through a learning[60] process over time. In the case of stage development, the revealed information would lead to possible scaling of the project, which will affect the level of investment and hence the decision to invest. Since this is not the case with this model, the process of resolving uncertainty in price[61] conflicts with the process of resolving uncertainty in reserve (which is through investment and exploitation of the resource). This conflict leads to earlier investment when reserve uncertainty is jointly considered with price uncertainty in the form of lower threshold investment price than that predicted by the constant reserve model.

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[60] See Copeland and Keenan (1998) Huchzermier and Loch (199) for more on learning options.

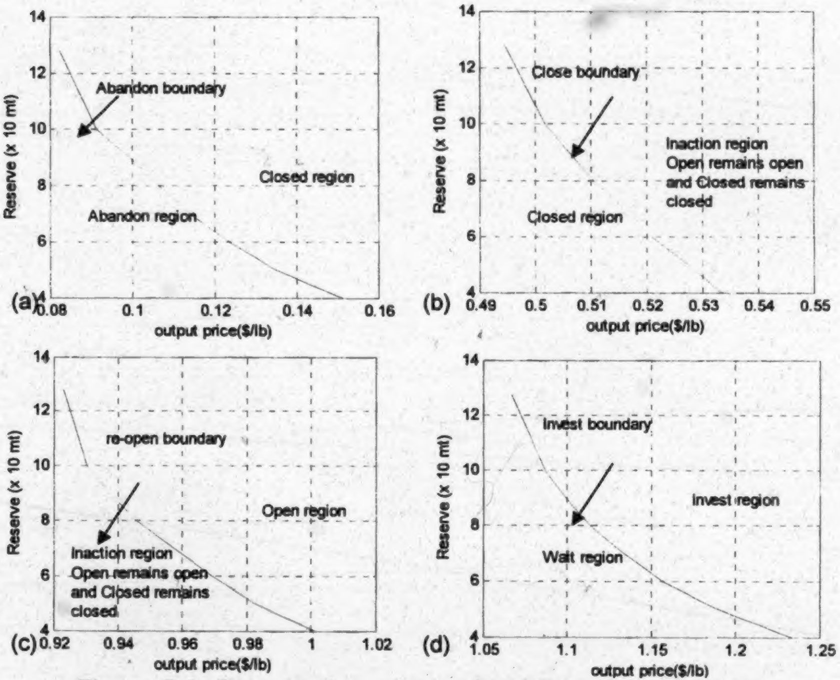

Figure 5-9: Exercise boundaries for different levels of reserve

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[61] This involves waiting on the investment decision until price is high enough to ensure profitability

Figure 5-9 shows the exercise regions for the 2DPR model. All boundaries have negative slope. Figure 5-9 (a) shows the boundary separating the closed and abandon regions. It illustrates that when price is low, reserve has to be considerably high in order to justify the continued closure of the mine at a cost otherwise the mine should be abandoned. Figure 5-9 (b) also illustrates that, at a lower price, a bigger mine (higher reserve) mine will delay closure relative to a smaller mine. Although, in this model, the same maintenance and closure costs were used for both low reserve and high reserve mines this trend is likely to be observed in practice. The decision to close an opened mine is partly influenced by the maintenance and closure cost; these two costs have multiplicative effect on the exercise boundary. Higher maintenance cost will trigger late closure and high closure cost will also delay closure. Bigger (higher reserve) mines are likely to incur high maintenance and reopening cost than smaller mines. An empirical study by Moel and Tufano (2002) found evidence that gold mines in North America that never closed (between 1988 and 1997[62]) had their fixed cost (proxy for maintenance cost) about 50% higher than mines that shut down at some point for economic reasons. Also higher reserve mines are more likely to remain open; in fact mines that closed between 1988 and 1997 had reserves about 45% less than mines that never closed (Moel and Tufano (2002)).

Figure 5-9 (c) again shows that, in the event of economic recovery, closed mines that have higher reserves will reopen earlier than smaller mines. Although in this case maintenance cost and reopening cost have offsetting effect on the exercise boundary this trend is also likely to be observed in practice. High maintenance costs (likely bigger mine) will trigger early reopening but higher reopening cost (also likely bigger mine) will delay reopening. The empirical study by Moel and Tufano (2002) did not find evidence for reopening and closure costs affecting reopen and close decisions.

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[62] Gold price started on the decline in 1997

Figure 5-9 (d) shows how investment decision changes with reserve. It indicates that higher reserve mine are developed earlier than in smaller reserve mines. For this model the same development cost was used for both high and low reserve so this trend is to be expected. This trend might not necessarily be observed in practice since in general the level of investment required directly depends on the size of the deposit. Also at the same capital requirement the investment decision is seen to be more sensitive to the level of reserve than the close, open or abandon decisions. This is likely to be observed in practice since the initial capital requirement is often several multiples of the cost involved to shutdown or reopen a mine.

Figure 5-10 shows how the exercise region behaves with uncertainty in reserve. The close boundaries in Figure 5-10(b) and the abandonment boundaries in Figure 5-10(a) suggest that higher volatility in reserve shrinks the optimal region that mines could remain closed. This as a result of later closure of operating mines (lower closure threshold prices) and early abandonment (higher threshold abandonment price) for both high and low reserve mines.

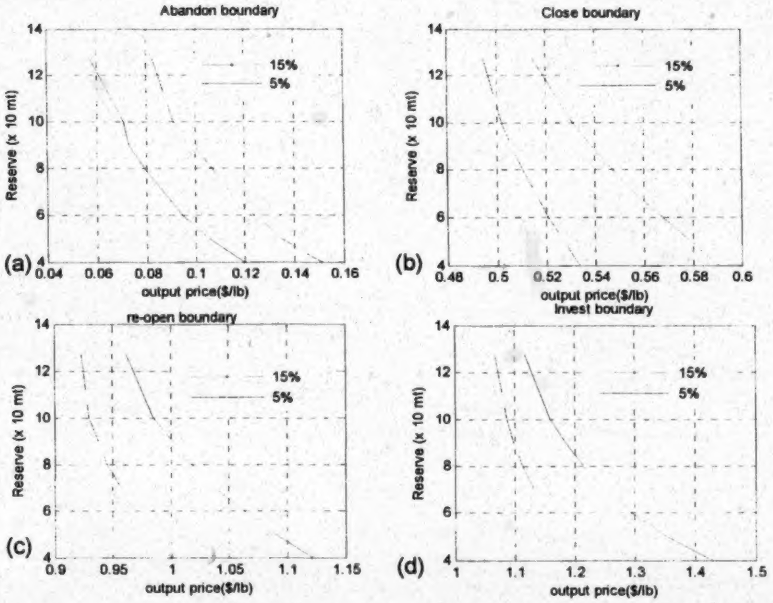

Figure 5-10: Behavior of exercise region with reserve volatility.

Figure 5-10 (c) and (d) also suggest that higher volatility in reserve affects the reopening and investment decisions of both smaller and larger mines in the same way. Operating mines remain opened longer, closed mines reopen earlier and there is earlier investment. Regarding early investment, it is probably the case that since the required investment often depends on the level of reserve, this trend may not necessarily be observed in practice. If the difference in capital requirement is high, then higher uncertainty in reserve may in fact delay larger reserve projects than smaller ones. This is likely to be observed in practice especially if the smaller reserve project is owned by a larger operating company.

Table 5-2: Exercise Boundaries for Commodity Price ($/lb); CRM and 2DPR

	2DPR	CRM
Abandon threshold, S_a*	0.0824	0.0561
Close threshold, S_c*	0.4948	0.5633
Re-open threshold, S_o*	0.9231	1.1048
Investment threshold, S_i*	1.0671	1.2922

5.6 Summary and Conclusions

Besides output price risk, natural resource firms are also faced with enormous risk due to the uncertainty in the level of reserve of the property they operate or wish to acquire. In order to account for reserve risk in valuing an asset, it is important to know the stochastic process that characterises the reserve process. The reserve process was assumed to follow a standard geometric Brownian motion process with both a drift and volatility term. This process is appropriate in the sense that for a matured and fully explored property one does not expect any extraordinary information during the operation of the property. The underlying commodity price was also assumed to follow a standard geometric Brownian motion and a 2D price-reserve (2DPR) model developed from the two stochastic processes.

The equilibrium partial differential equation that governs the value of a mine was developed under equilibrium conditions and the assumption of the capital asset pricing model (CAPM). For a given state of the mine, the governing equilibrium equations were formulated subject to the appropriate terminal and boundary conditions. The results from the 2DPR model were compared with that from the equivalent constant reserve model (CRM) and NPV.

The results show that generally within the region where there is the flexibility to exercise the close or abandon option, the 2DPR model assigns a higher asset value than conventional NPV. The value from flexibility is thus higher than the loss in value due to uncertainty in reserve. Comparing results from 2DPR to that from CRM show that using a constant reserve assumption to value a mineral property when there is uncertainty in the level of reserve in general assigns higher value to the asset. This is especially so when the expected growth rate in reserve is null. However, if the expected growth rate in reserve is non-zero then depending on the growth rate the asset may be undervalued by using a constant reserve assumption.

Reserve uncertainty generally decreases the value of the asset and decreases the value of the firm's investment option on the asset even more. This generally leads to earlier investment than for the case of constant reserve. The 2DPR model also predicts delayed closure of opened mines but early opening of closed mines than that from CRM. These results stem from how uncertainty in reserve is resolved over time. Whilst uncertainty in price risk is resolved through waiting and doing nothing, uncertainty in reserve is resolved only by investing and exploiting the deposit. These two means of uncertainty resolution tend to offset each other in the model presented. 2DPR however predicts early mine abandonment than CRM.

6 CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

Commodity prices are often characterised by high fluctuations and as a result, mineral resource industry unlike other industries could have large variations in revenues. Besides market risks from unstable prices, resource firms are also faced with the uncertainty in the quantity and quality of reserve in place. A profitable company could be at the brink of bankruptcy just by investing in a wrong project since mineral investments are usually capital-intensive and irreversible. Investment decisions must therefore be based on rigorous analysis taking into account these specific risks.

Conventional capital budgeting methods like the discounted cash-flow, one-period capital asset pricing (and other variations) are good investment decision tools when there is perfect and certain information about the value drivers regarding the project. They are also good tools when the decision is completely reversible without any losses. In the presence of uncertainty and irreversibility, if there is also flexibility in the course of actions then these conventional methods lack the capabilities to handle the interactions of the various decisions and hence the investment decision.

Derivative mine valuation or real options is able to deal with the limitations of conventional methods. It is able to incorporate the risk 'structure' of the value drivers as well as any operating or managerial flexibility at the disposal of management prior to and after the investment decision. Previous mineral resource valuation models account for the uncertainties in commodity prices but treat the reserve base as known and constant. This limits the applicability of these models since the uncertainty in reserve is a peculiar characteristic of the resource industry.

The objective of this thesis was therefore to develop a mineral valuation model that takes into account the uncertainty associated with reserve and examine how the inclusion of reserve uncertainty affects mine value, as well as, mine operation and investment decision. Two models have been developed to capture the effect of uncertainty in reserve (in addition to uncertainty in commodity price).

The 2D price model (2DPM) assumes that the reserve base is known but then market condition (commodity price) reduces this level by imposing a constraint on the economic cut-off grade. The uncertainty in reserve is thus due to the uncertainty associated with output price. The second model, 2D price-reserve model (2DPR), attributes the uncertainty in reserve to both changes in grades and price as well as endogenous reserve additions. Endogenous reserve additions is assumed to be as a result of expected changes in reserve from exploration activities during mining to convert indicated resources to mineral reserve. The two models were applied to a hypothetical mine and the results are compared with those from using the constant reserve model (CRM) and the net present value (NPV). A graphical user interface has also been developed in order to facilitate the easy use of the models. This is the first effort towards the development of a GUI environment to advance mineral investment valuation using option pricing theory.

The reserve variation is modeled as a function of the changes in commodity price and thus results in a one-factor model with cut-off grade flexibility. If the change in cut-off grade in response to price changes is exogenous to the model (2DPM), the reserve process can be characterised by a Weiner process with a drift term η which could either be less or greater than the drift (physical depletion, q) in the constant reserve model (CRM). The stochastic component is proportional to the sensitivity of the reserve-price curve and is completely defined by the Weiner process that characterises price.

From the detailed simulation and analysis of the results from 2DPM, CRM and NPV, the following conclusions are drawn:

- This is the first research study, which focuses on price and reserve uncertainties in derivative mine valuation.

- Constant reserve model generally predicts higher asset values than the corresponding 2DPM model with the same level of reserve as potential reserve. At a copper price of $1.5/lb 2DPM values a 127.8mt mine at $996million about 138million (12%) less than that by CRM.
- Generally, when output prices are within $0.65/lb (marginal cost) there is the flexibility to exercise the close or abandon options so the 2DPM model assigns a higher asset value than conventional NPV. At prices slightly higher

than marginal cost ($0.782/lb) however, NPV assigns higher values. At a price of $0.7/lb NPV values a 127.8mt mine at $194 million whilst 2DPM values it at $235million.

- 2DPM compared with CRM with a lower level of reserve (but higher quality reserve) indicate that an ad-hoc flexible cut-off grade management as presented may not necessarily add value to the asset in question. At copper price of $1.4/lb a 99.2mt mine at an average grade of 0.69% of copper is valued at $905million by CRM whilst 2DPM values this mine with cut-off grade flexibility and potential reserve of 127.8mt at $879million.
- Cut-off grade policy is important in determining the value of a mine. Cut-off grade determination should not be made in isolation without consideration of optimal operating policy. This is especially so when one is dealing with vein deposits or deposits with higher grade variability.
- Deposits with low variability in grade may be undervalued if the potential reserve is not taken into account during valuation
- 2DPM predicts that operating mines will close earlier and closed mines will re-open earlier compared to CRM.
- Lower project values from 2DPM require that mineral assets be abandoned earlier than assets that are valued using CRM. At an annual maintenance cost of $0.5million CRM abandons the mine at a copper price of $0.05/lb whilst 2DPM abandons it at $ 0.07/lb.
- Under ad-hoc flexible cut-off grade policy, such as, the one prescribed in this model, firms may wait longer than necessary to invest in mineral opportunities

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especially if the deposit has higher variability in grade. At an investment cost of $150million, CRM suggests that a 127.8mt project should come upstream when copper price hits $1.29/lb whilst 2DPM suggests a price of $1.41/lb.

- Unlike CRM, increase in volatility of commodity price generally leads to a decrease in value of the mine using 2DPM which is different from the general notion of effect of higher volatility on option value.

By assuming that reserves follow a geometric Brownian motion and not in particular dependent on price (2DPR), the resulting equilibrium equation that governs the mine was found to be an elliptic partial differential equation. The results from the 2DPR model compared with those from the equivalent constant reserve model (CRM) and NPV indicate that:

- Generally, within the region where there is the flexibility to exercise the close or abandon option, the 2DPR model assigns a higher asset value than conventional NPV. At a price of $0.7/lb NPV values a 127.8mt mine at $194 million whilst 2DPR values it at $218million.

- CRM generally assigns higher value to the asset. This is especially so when the expected growth rate in reserve is null. For the model parameters used, CRM values a 127.8mt mine at a premium of about 16%.
- If the expected growth rate in reserve is non zero, then depending on the growth rate, the asset may be undervalued for a constant reserve assumption. With a reserve volatility of 15%, 2DPR values are higher than

CRM values if annual growth rate in reserve is estimated to be greater than 2%.

- Reserve uncertainty generally decreases the value of mineral assets and decreases the value of the firm's investment option on the asset even more. The increase in uncertainty leads to earlier investment than in the case when reserve is considered as constant. CRM predicts that, a threshold copper price of $1.29/lb is high enough to warrant capital expenditure of $150million to develop a 127.8mt deposit. 2DPR however predicts a threshold price of $1.07/lb.
- 2DPR also predicts delayed closure of opened mines but early opening of closed mines than CRM.
- Lower value from 2DPR predicts early abandonment of mineral assets.
- Higher uncertainty in reserve decreases the value of a mineral asset again different from the general view of the effect of increase volatility on the value of options.
- Higher uncertainty in reserve also leads to earlier investment, earlier reopening late closure and early abandonment.

6.2 Recommendations for Future Research

Cut-off grade policy is important in determining the value of a mine. The cut-off grade policy presented in the 2DPM was exogenous to the model and was not necessarily an optimal one. A possible extension of the 2DPM will be a model that endogenously determines the optimal cut-off grade policy just as optimal operating policy. This requires a 3-D model (with price, reserve and cut-off grade as the state variables) and may require the use of simulation as the solution procedure since the solution using either finite element or finite difference is usually not easily tractable once the state variables increase beyond two.

The stochastic process used to model the underlying variables is important in determining asset value. Another extension of the 2DPM model will be to use a mean reverting process for price while at the same time determining cut-off grade operating policy endogenously. An extension of the 2DPR model in this respect will be to use a mixed jump and diffusion process to model the reserve process. This mixed process will be able to capture the possibility extraordinary discovery during operation.

The 2DPR model assumed that the same level of investment, as well as, the same per unit cost was required for different levels of reserve. Larger mines generally require higher investment capital and operate at lower marginal cost. Another extension of the 2DPR model will be to take these into account. A possible model might be one that models investment and operating costs as some piecewise function of reserve and not necessarily stochastic.

The uncertainty in reserve was assumed to be constant in the 2DPR model. This uncertainty in reserve however decreases over time as mining progresses. An extension of the 2DPR model could be a model that captures the declining characteristics of reserve uncertainty with time.

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A Appendix A: CTSP SOFTWARE INTERFACE

A.1 Introduction

The fundamental understanding of the continuous-time stochastic process underlying option theory has been a limiting factor against its application in industry. Industry has therefore continued to use the simplest DCF methods notwithstanding their short-comings. The objective in this Chapter is to introduce a graphical user-interface that has been developed to facilitate the use of the three valuation models discussed in this thesis.

The fundamental power of graphical user interface (GUI) is that it provides a means through which individuals can communicate with the computer software without the need for programming commands. A GUI is therefore only as good as the assumptions that underlie the models that it facilitates in the evaluation process. The user of any GUI that is based on a model must therefore seek to understand the model assumptions and make a decision if the model assumptions conform to their specific circumstance. The model assumptions to this GUI are in Chapters 4 and 5 of this thesis. The use of this GUI requires the installation of both FEMLAB (version 3.1i or higher) and MATLAB (version 7 or higher).

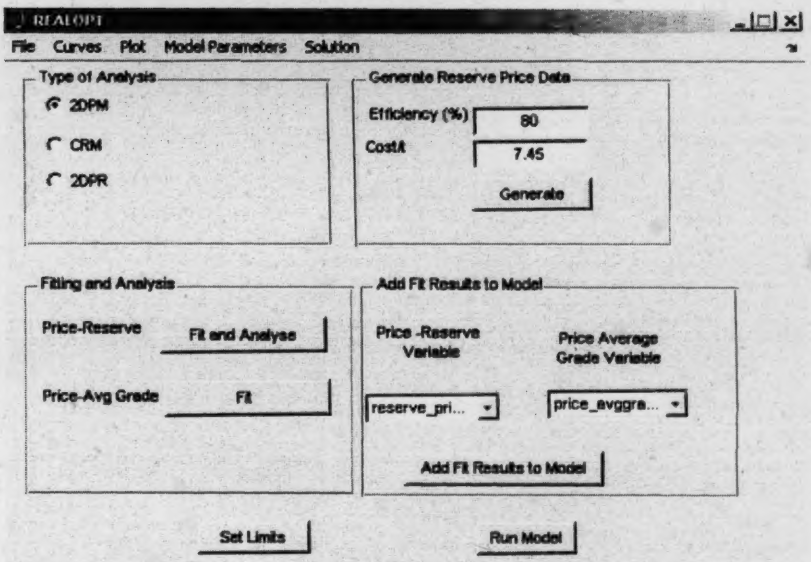
MATLAB was chosen as the platform for building the interface because it has a preeminent computing environment, provides quick access to many data processing functions and toolboxes and easily allows one to create special purpose applications to be used by others. It is also seamlessly compatible with FEMLAB which is the driving engine for solving the model equations.

A.2 General Overview

The developed GUI allows the user to select which type of valuation analysis to be performed for a given project valuation problem. In a given GUI environment, the user enters the model parameters that are associated with the selected type of analysis. These parameters include economic, technical and other project specific data. When carrying out a 2DPM analysis, the user has to fit the right estimation model to the reserve-price and price-average grade curves. This requires familiarity with simple goodness-of-fit statistics in order to select the best fitting model. The software allows the user to do multiple valuation analyses and compare the results to one another through postprocessing of the results.

A.3 Building a 2DPM model

The first step in building a 2DPM model is to define the reserve-price curve. This is defined from the tonnage-grade curve. This tonnage-grade data should be in the text format and arranged in the order of cut-off grade, tonnage above cut-off and average grade. This data is added to the model by selecting ***New*** on the ***File*** menu and selecting ***DPM***. The reserve-price and price-average grade curves are generated from this data by clicking on the ***Generate*** button. The production cost per ton, as well as, the overall processing efficiency must be entered before the generation of these curves.


REALOPT
File
Curves
Plot
Model Parameters
Solution
Type of Analysis
Generate Reserve Price Data
Generate
Fitting and Analysis
Price-Reserve
Fit and Analyse
Price-Avg Grade
Fit
Add Fit Results to Model
Price -Reserve Variable
Price Average Grade Variable
Add Fit Results to Model
Set Limits
Run Model

● 2DPM
○ CRM
○ 2DPR

Efficiency (%) 80
Cost/t 7.45

reserve_pri...
price_avggra...

Figure A-1: Main User Interface

The 2DPM model requires the reserve-price sensitivity curves as inputs to the model. The generated reserve-price data have to be fit with an appropriate model. The ***Fit and Analyze*** button is used to fit a model to the reserve-price data and determine the associated sensitivity curves. Clicking on the ***Fit and Analyze*** button opens the curve fitting toolbox with the reserve-price scatter plot as shown in Figure A-2. The ***Fit*** button is also used to fit the price-average grade curves. Fitting and analysis are done using MATLAB's inbuilt fitting toolbox.

The Curve Fitting Tool provides several features that facilitate data and fitting analysis. Clicking on the ***Data, Fitting, Exclude, Plotting,*** or ***Analysis*** button opens the associated GUI. These associated GUIs are described below. The ***Data*** button allows one to import and create dataset and remove any outliers from the dataset. The ***Fitting*** button in Figure A-2 is used to fit parametric and non parametric models to the data, examine and compare the fit results including fitted coefficient values and goodness-of-fit statistics. The ***Exclude GUI*** allows you to create exclusion rules for a data set. An exclusion rule identifies data to be excluded when fitting a model. The excluded data can be individual data points, or a section of predictor or response data. The ***Analysis*** button allows one to evaluate (interpolate or extrapolate), differentiate, or integrate a fit, plot the analysis results and the data set[63].

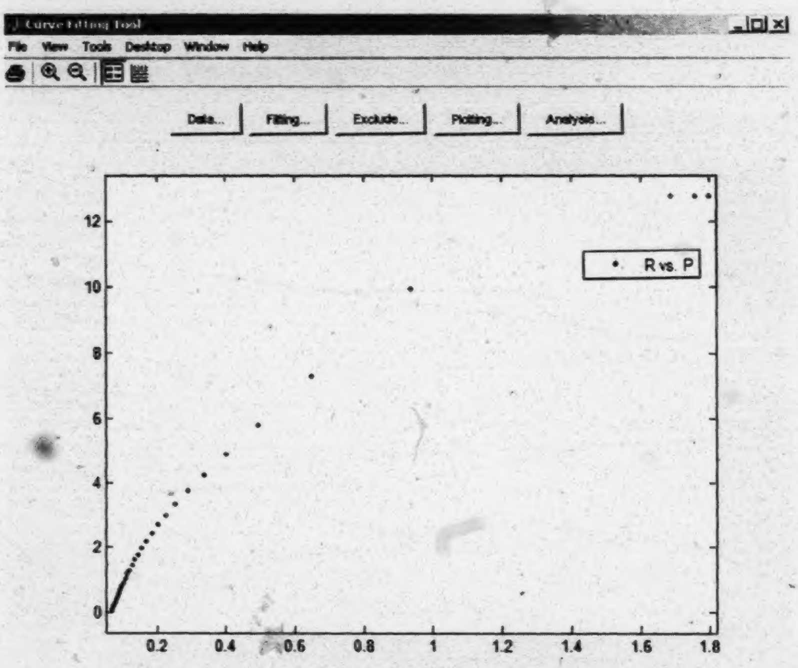

Curve Fitting Tool
File
View
Tools
Desktop
Window
Help
Data...
Fitting...
Exclude...
Plotting...
Analysis...
R vs. P
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

Figure A-2: Curve Fitting Interface

---

[63] For more information on these buttons see MATLAB documentation

A.3.1 Determining the Best Fitting Model

To determine the best curve that fits the data, it is important to examine both the graphical and numerical results. The initial approach in determining the best curve should always be a graphical examination of the fits and residuals. The residuals and prediction bounds are graphical measures, while the goodness-of-fit statistics and confidence bounds are numerical measures. Generally speaking, graphical measures are more beneficial than numerical measures because they allow the entire data set to be viewed at once, and they can easily display a wide range of relationships between the fitted model and the data. Numerical measures are more narrowly focused on a particular aspect of the data and often try to compress that data into a single number (MATLAB 2004).

The graphical fit to the price-reserve curve, shown in Figure A-3, indicates that all the fit equations with the exception of the cubic interpolation (*interpol1)* seem to fit the data well especially at lower values of price. The residual from a fitted model is defined as the difference between the response data and the fit to the response data at each predictor value, as shown in equation (A-1).

$$r = y - \hat{y} \qquad \text{(A-1)}$$

The residuals are plotted by selecting ***Residuals*** from the ***View*** menu item in Figure A-2. The residuals approximate the random errors. Therefore, if the residuals appear to be randomly scattered about zero, it suggests that the model fits the data well. However, if the residuals display a systematic pattern, it is a clear sign that the model poorly fits the data. A look at the residuals shown in Figure A-3 therefore suggests that the 4th and 5th degree polynomials do not fit the data well. These fitted models can be deleted from the scatter plot by using the ***Plotting*** GUI as shown in Figure A-4.

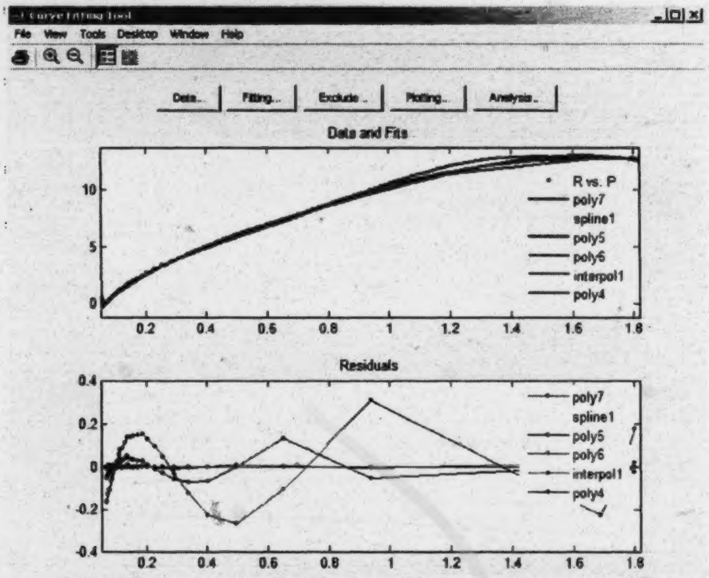
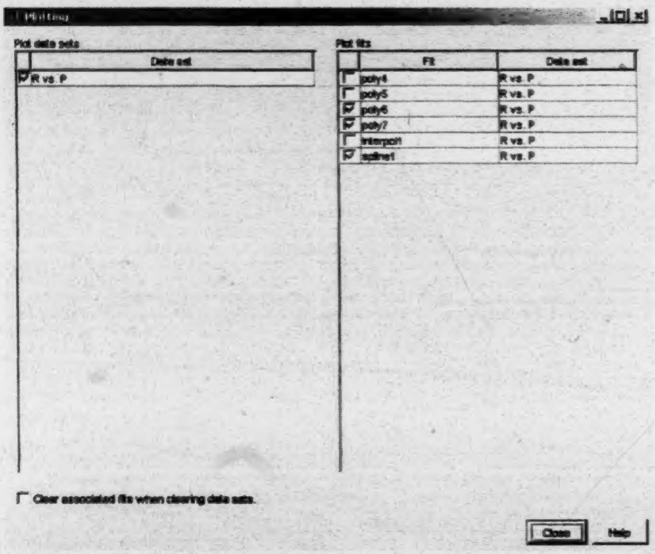

Curve Fitting Tool
File
View
Tools
Desktop
Window
Help
Data...
Fitting...
Exclude...
Plotting...
Analysis...
Data and Fits
R vs. P
poly7
spline1
poly5
poly6
interpol1
poly4
10
5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Residuals
poly7
spline1
poly5
poly6
interpol1
poly4
0.4
0.2
0
-0.2
-0.4
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

Figure A-3: Scatter plot of reserve-price fitted with several models

After using graphical methods to eliminate poor fits, the goodness-of-fit statistic is further examined to determine which model best fits the data. There are two types of numerical fit results displayed in the Fitting GUI: goodness-of-fit statistic and confidence intervals on the fitted coefficients. The goodness-of-fit statistic helps to determine how well the curve fits the data. The confidence intervals on the coefficients determine their accuracy.



Plot data sets

	Data set
☑	R vs. P

Plot fits

	Fit	Data set
☐	poly4	R vs. P
☐	poly5	R vs. P
☑	poly6	R vs. P
☑	poly7	R vs. P
☐	interpol1	R vs. P
☑	spline1	R vs. P

☐ Clear associated fits when clearing data sets.

Figure A-4: Plotting GUI used to remove interpol1, poly4 and poly5

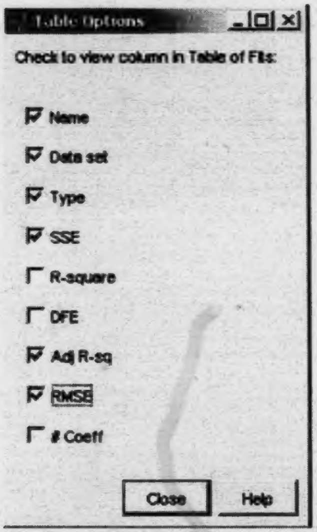
The goodness of fit statistics that are supported for parametric models are:[64]

- The sum of square error (SSE)
- R-square
- Adjusted R-square
- Root mean squared error (RMSE)

The fit statistic of interest can be selected from the ***Table Options*** in the ***Fitting*** GUI in Figure A-6. The fit statistics are displayed in the ***Results*** list box in the ***Fit Editor***. For all fits in the current curve-fitting session, the goodness-of-fit statistics can be compared in the ***Table of fits***. Low RMSE and SSE and high R-Square and adjusted R-square indicate good fit. The opposite measures indicate a poor fit.

---

[64] See Section B8 in Appendix B for the definition of these fit statistics

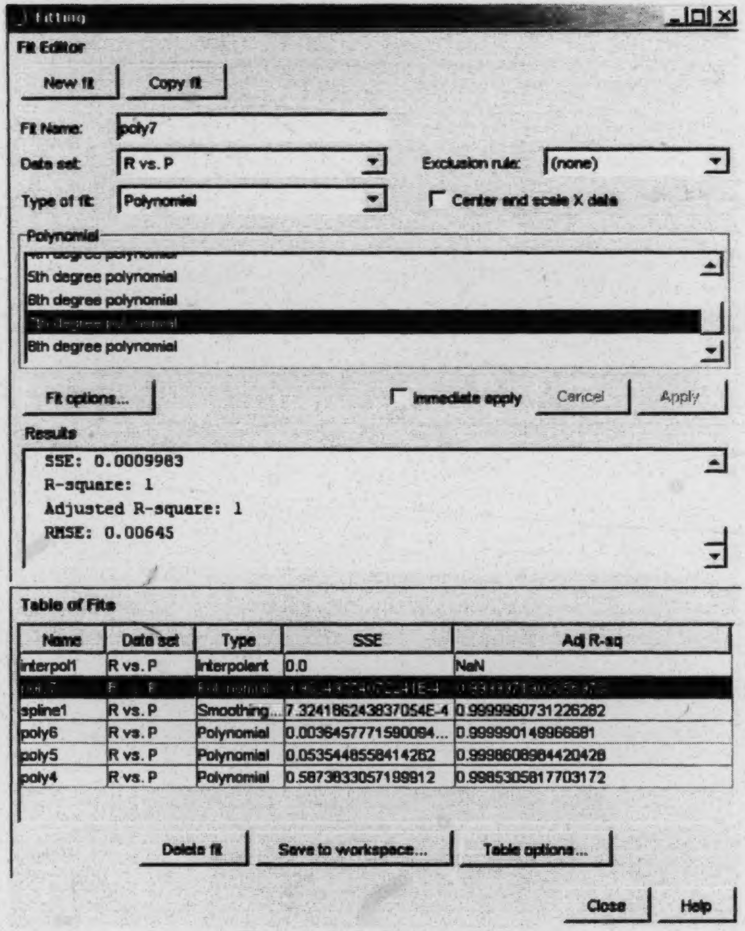


Check to view column in Table of Fits:

☑ Name
☑ Data set
☑ Type
☑ SSE
☐ R-square
☐ DFE
☑ Adj R-sq
☑ RMSE
☐ # Coeff

Figure A-5: Goodness of fit statistic options

The SSE goodness-of-fit statistic shown in Figure A-6 suggests that the cubic spline (*interpol1)* is the best model but this model was deleted by the graphical examination. The 7th degree polynomial is therefore selected as the best fit for reserve-price data.



Fitting
Fit Editor
New fit
Copy fit

Fit Name: poly7

Data set: R vs. P

Exclusion rule: (none)

Type of fit: Polynomial

☐ Center and scale X data

Polynomial
5th degree polynomial
6th degree polynomial
7th degree polynomial
8th degree polynomial
Fit options...
☐ Immediate apply
Cancel
Apply
Results

```
SSE: 0.0009983
R-square: 1
Adjusted R-square: 1
RMSE: 0.00645
```

Table of Fits

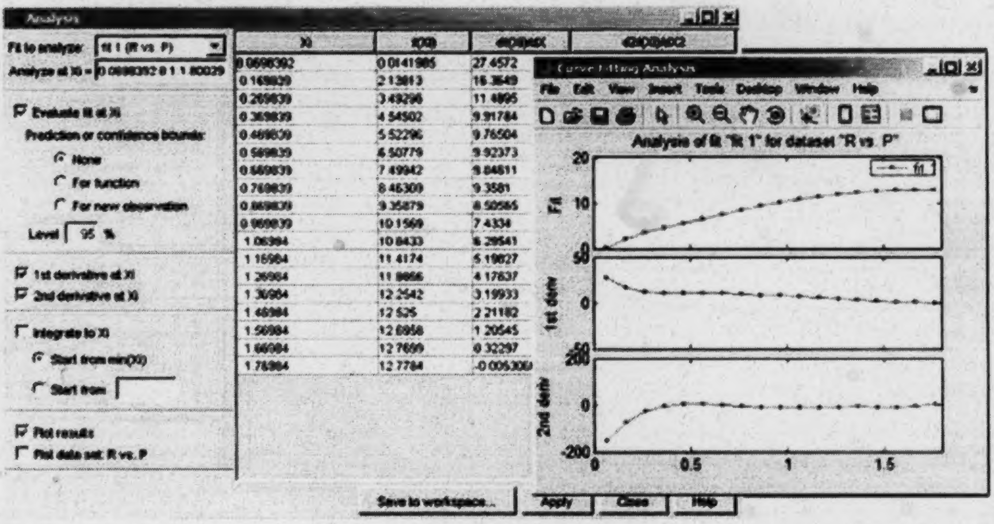
Name	Data set	Type	SSE	Adj R-sq
interpol1	R vs. P	Interpolant	0.0	NaN
poly7	R vs. P	Polynomial	[illegible]	[illegible]
spline1	R vs. P	Smoothing...	7.324186243837054E-4	0.9999960731226282
poly6	R vs. P	Polynomial	0.0036457771590094...	0.999990149966681
poly5	R vs. P	Polynomial	0.0535448558414262	0.9998608984420428
poly4	R vs. P	Polynomial	0.5873833057199812	0.9985305817703172

Delete fit
Save to workspace...
Table options...
Close
Help

Figure A-6: Fitting GUI

A.3.2 Determining the first and second numerical derivatives

The ***Analysis*** GUI is used to determine the price-reserve sensitivity curves. This GUI is shown in Figure A-7 and is opened by clicking on the ***Analysis*** button in Figure A-2. The derivatives are determined by defining the analysis interval, checking the ***evaluate fit at Xi***, ***1st derivative at Xi***, *and* ***2nd derivative at Xi*** checkboxes and clicking on the ***Apply*** button. The ***Plot results*** checkbox should also be checked to have a visual sense of the fit and the numerical derivatives. The results should be saved by clicking on the ***save to workspace*** button. When analyzing the price-average grade data it is not necessary to check the ***1^{st} and 2^{nd} derivative*** checkboxes. The analysis interval does not have to be the same for the two analyses.



Fit to analyze: fit 1 (R vs. P)

Analyze at Xi = 0.0698392:0.1:1.80029

☑ Evaluate fit at Xi

Prediction or confidence bounds:

● None

○ For function

○ For new observation

Level 95 %

☑ 1st derivative at Xi

☑ 2nd derivative at Xi

☐ Integrate to Xi

● Start from min(Xi)

○ Start from

☑ Plot results

☐ Plot data set: R vs. P

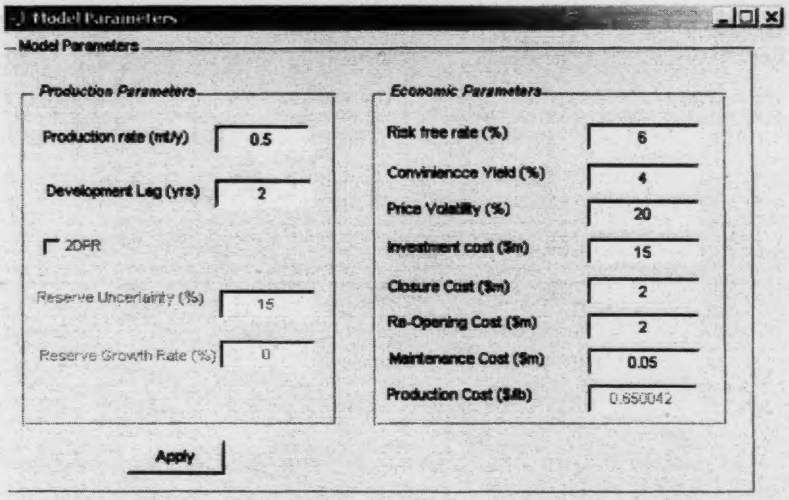
Xi	f(Xi)	df(Xi)/dX	d2f(Xi)/dX2
0.0698392	0.0141985	27.4572	
0.169839	2.13813	16.3649	
0.269839	3.49296	11.4895	
0.369839	4.54502	9.91784	
0.469839	5.52296	9.76504	
0.569839	6.50779	9.92373	
0.669839	7.49942	9.84611	
0.769839	8.46300	9.3581	
0.869839	9.35879	8.50565	
0.969839	10.1569	7.4334	
1.06984	10.8433	6.29541	
1.16984	11.4174	5.19827	
1.26984	11.8856	4.17637	
1.36984	12.2542	3.19933	
1.46984	12.525	2.21182	
1.56984	12.6958	1.20545	
1.66984	12.7699	0.32297	
1.76984	12.7784	-0.005300	

Analysis
Curve Fitting Analysis
File Edit View Insert Tools Desktop Window Help
Analysis of fit "fit 1" for dataset "R vs. P"
fit
Fit
20
10
0
1st deriv
50
0
-50
2nd deriv
200
0
-200
0
0.5
1
1.5
Save to workspace...
Apply
Close
Help

Figure A-7: Analysis GUI

The results from the analysis are added to the 2DPM model by selecting the appropriate variable names and clicking on the ***Add fit results to model*** button on the main GUI interface in Figure A-1. The data set added to the model is used to create a cubic spline interpolation function in FEMLAB to define the sensitivity curves.

Other model parameters are specified on the GUI shown in Figure A-8 by clicking on the ***Model Parameters*** menu on the main GUI in Figure A-1. The limits of the solution domain are set by clicking on the ***Set Limit*** button. For 2DPM it is not required to set the maximum reserve limit as it is defined by the tonnage curve. The solution is obtained by clicking on the ***Run model*** button. Building a CRM or 2DPR model only requires the specification of model parameters and setting the limits of the solution domain.



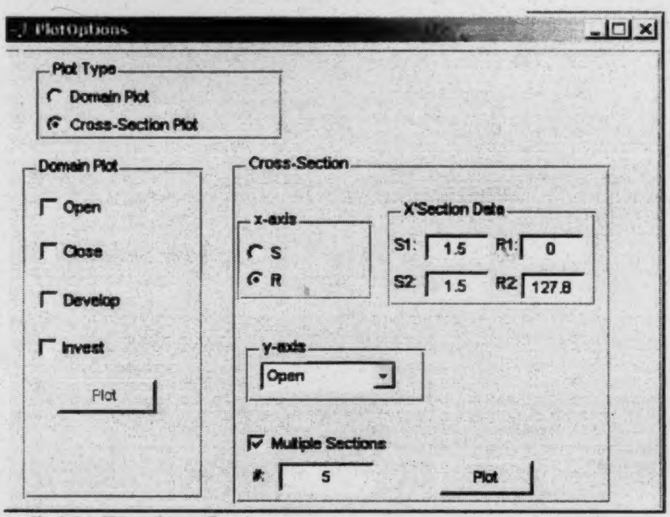
Production rate (mt/y) 0.5
Development Lag (yrs) 2
☐ 2DPR
Reserve Uncertainty (%) 15
Reserve Growth Rate (%) 0

Risk free rate (%) 6
Convinience Yield (%) 4
Price Volatility (%) 20
Investment cost ($m) 15
Closure Cost ($m) 2
Re-Opening Cost ($m) 2
Maintenance Cost ($m) 0.05
Production Cost ($/lb) 0.650042

Figure A-8: Model Parameters Interface

A.4 Postprocessing of Results

The results can be seen by clicking on the ***Solution*** menu and selecting the appropriate analysis type. By default it reports the results for the maximum level of reserve that is specified. The user can however get the results for other levels of reserve by taking a cross-section through the solution domain. There is also the option to display the solution over the whole solution domain as a contour plot. This is achieved by selecting the menu item ***more....*** on the ***Solution*** menu and specifying plot parameters shown in Figure A-9. The resulting plots are MATLAB figures so the user can change the figure properties by toggling through the menu items of the figure.



PlotOptions

Plot Type
○ Domain Plot
● Cross-Section Plot

Domain Plot
☐ Open
☐ Close
☐ Develop
☐ Invest

Cross-Section
x-axis
○ S
● R

X'Section Data
S1: 1.5 R1: 0
S2: 1.5 R2: 127.8

y-axis
Open

☑ Multiple Sections
#: 5

Figure A-9: Available Plotting Options

A.5 Summary and Conclusions

This chapter discussed the graphical user-interface that facilitates the use of the two developed models, as well as, the constant reserve model. The interface is very simple and easy to use without requiring the user to understand the Mathematical underpinnings of real option valuation. The use of this GUI however requires the user to have at least both FEMLAB (3.1i) and MATALB (7) installed and this is regarded as a major limitation of the GUI as it is not stand-alone.

B Appendix B

B.1 Analytical Solution to the Futures Model

From Feynman Kac's stochastic representation formula if F is a solution to the PDE of the form given in equations (B-2) and (B-3) then S satisfies equation (B-4) and the solution of F is given by equation (B-5).

$$F_t(S,t)+\lambda(S,t)F_S(S,t)+\frac{1}{2}\beta^2(S,t)F_{SS}(S,t)-\gamma(S,t)F(S,t)=0 \qquad \text{(B-2)}$$

Subject to:

$$F(S,T)=\Phi(S_T) \qquad \text{(B-3)}$$

$$dS=\lambda(S,t)dt+\beta(S,t)dw \qquad \text{(B-4)}$$

$$F(S,t)=E_t\left[\Phi(S_T)\exp(-\int_t^T\gamma_s ds)\right] \qquad \text{(B-5)}$$

In this regard the solution to the equations (B-6) and (B-7) is given by equation (B-8) and the price process satisfies equation (B-9)[65]. The solution to the stochastic differential equation in (B-9) is given in equation (B-10).

$$\frac{1}{2}F_{SS}\sigma^2S^2+F_S(r-\delta)S-F_\tau=0 \qquad \text{(B-6)}$$

---

[65] This is the price process under an equivalent risk neutral probability measure.

Subject to

$$F(S,T) = S \tag{B-7}$$

$$F(S,t) = E_t(S_T) \tag{B-8}$$

$$dS = (r - \delta)Sdt + \sigma Sdw \tag{B-9}$$

$$S_T = S_0 e^{(r-\delta)T} \times e^{(-\frac{\sigma^2}{2}T+\sigma w(T))} \tag{B-10}$$

$$E_0(S_T) = S_0 e^{(r-\delta)T} E_0[e^{-\frac{\sigma^2}{2}T+\sigma(w_T)}] = S_0 e^{(r-\delta)T} \tag{B-11}$$

The expression $e^{-\frac{\sigma^2}{2}T+\sigma(w_T)}$ can be shown to have an expectation equal to 1; this is a property of Martingales.

B.2 Ito's Lemma

Ito's Lemma is the chain rule equivalent of derivatives (or the Taylor series approximation) when dealing with stochastic variables. When dealing with deterministic variables, second order terms can be set to zero when changes in the state variables are small but this is however not the case when dealing with stochastic variables. This has to do with the property of the normal distribution. If F is a function of *x* and *y* then equations (B-12) and (B-13) are the differential of F if *x* and *y* are deterministic and stochastic respectively.

$$dF = F_x dx + F_y dy \tag{B-12}$$

$$dF = F_x dx + F_y dy + \frac{1}{2} F_{xx} (dx)^2 + F_{xy} dydx + \frac{1}{2} F_{yy} (dy)^2 \qquad \text{(B-13)}$$

B.3 The Finite Element Approximation

The basic idea of finite element is to approximate the solution of a given differential equation of the form shown in equation (B-14) with an approximate function, $\bar{u}$, which is set of algebraic sum of simple functions as shown in equation (B-15). $L(\cdot)$ in equation (B-14) is a differential operator and *u* assumed to be a single dependent variable in two space dimension. These simple functions ϕ_is s are called the *basis, shape or interpolating* functions as they form the basis of the descretization process. The weights a_is are found such that the residual function defined in equation (B-16) is minimal. This is accomplished by solving the system of equations defined by equation (B-17). The w_js are arbitrary weighting functions and dA is the elemental area. The nature of the weighting function determines the framework of the finite element approximation method. The most common framework is the *Galerkin* method where the weighting function is set equal to the basis function; thus $w_i = \phi_i$.

$$L(u) - f = 0 \qquad \text{(B-14)}$$

$$\bar{u} = \sum_{i=1}^{N} a_i \phi_i \qquad \text{(B-15)}$$

$$R = L(\bar{u}) - f \qquad \text{(B-16)}$$

$$\int_{\Omega} w_j R dA = 0; \forall j = 1,.....N \qquad \text{(B-17)}$$

Substituting (B-15) and (B-16) into equation (B-17) reduces to equation (B-18). If the differential operator is linear then equation (B-18) reduces to equation (B-19)

$$\int_{\Omega} L\left(\sum_{i=1}^{N} a_i \phi_i\right) w_j dA = \int_{\Omega} f w_j dA \quad \forall j = 1,.....N \qquad \text{(B-18)}$$

$$\sum_{i=1}^{N} a_i \underbrace{\int_{\Omega} L(\phi_i) w_j dA}_{K_{ij}} = \underbrace{\int_{\Omega} f w_j dA}_{F_j} \quad \forall j = 1,.....N \qquad \text{(B-19)}$$

$$\begin{pmatrix} k_{11} & \cdots & k_{1N} \\ \vdots & \ddots & \vdots \\ k_{N1} & \cdots & k_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ F_N \end{pmatrix} \qquad \text{(B-20)}$$

The finite element solution to the linear PDE reduces to solving a system of linear equations given by equation (B-20). Equation (B-20) is the strong formulation of the finite element problem. The *weak* formulation is often used in order to reduce the differentiability requirement on the trial and basis functions. This involves rewriting equation (B-18) by using integration by parts for 1D problems and greens formula for 2D problems. Using strong formulation requires that the test function be selected such that it satisfies the boundary conditions.

B.4 Finite element implementation in FEMLAB

B.4.1 Descritization of the solution domain:

This involves the division of the solution region into non-overlapping small elements of simple shapes; in 1D these elements are lines and in 2D are either triangles or quadrilaterals. In finite element terms, this is known as *mesh* generation. FEMLAB uses the Delaunay triangulation algorithm in MATLAB to generate these meshes. It has the capability to either generate *unstructured* mesh (triangular elements) or *mapped (structured)* mesh (quadrilateral elements). The use of mapped mesh requires the geometry to be fairly regular. The boundaries defined in the geometry are also partitioned into mesh edges, called *boundary elements* and set to conform to the triangles in the adjacent subdomain. It is possible to densify the mesh (mesh refinement) either locally through selective meshing in some regions of the domain or globally within the entire solution domain. The quality of the solution depends on the quality and density of the finite element. High mesh quality also enhances computational efficiency by decreasing the number of iterations required to achieve convergence (Edwini-Bonsu, 2004).

B.4.2 Descritization of Governing Equations

In coefficient form the general PDE is written in the form giving in equation (B-21).

$$\begin{cases} d_a \dfrac{\partial u}{\partial t} + \nabla \cdot (-c\nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + au = f & \text{in } \Omega \\ \mathbf{n} \cdot (c\nabla u + \alpha u - \gamma) + qu = g - h^T \mu & \text{on } \partial\Omega \\ hu = r & \text{on } \partial\Omega \end{cases} \tag{B-21}$$

The descretization of the above equations starts by approximating the solution u with a function that can be described by a finite set of parameters, the so called degrees of freedom (DOF), (FEMLAB 2005). This approximation is then substituted into the weak form of the equations to obtain a system of equations for the degrees of freedom. The dependent variables are thus expressed in terms of the degree of freedom as shown in equation (B-22) where the ϕ_is are the shape functions and the $\tilde{a}_i$s are the degrees of freedom. Each shape function is such that on the *i*th mesh interval it has a value equal to 1 at the *i*th node and zero at every other node. This is facilitated by the introduction of local coordinates (or element coordinates). Since ϕ_i has a value of 1 at the nodes, the solution at the nodal points is indeed he same as the degrees of freedom.

$$\tilde{u} = \sum_i \tilde{a}_i \phi_i \tag{B-22}$$

$\tilde{a}$ as the vector of DOFs with components $\tilde{a}_i$ is called the solution vector since their values completely characterize the nature of the trial function, $\tilde{u}$. Let $\tilde{v}$ be an arbitrary function called the test function ($\tilde{v}$ is not really arbitrary as it is required to belong to a suitably chosen well behaved class of functions (FEMLAB

2005). FEMLAB uses the *Galerkin* method and so sets the test functions $\tilde{v}$ to be the same as the finite elements.

The finite element statement for the PDE in equation (B-21) is given by equation (B-23).

$$\int_{\Omega} \tilde{v}\nabla \cdot (-c\nabla \tilde{u} - \alpha \tilde{u} + \gamma) dA = \int_{\Omega} \tilde{v}\left(f - \beta \cdot \nabla \tilde{u} - a\tilde{u}\right) dA \quad \text{in } \Omega \tag{B-23}$$

Using greens formula equation (B-23) can be written as equation (B-24) where *ds* is the length of the element and **n** is the unit normal vector to the boundary.

$$\int_{\partial\Omega} \tilde{v} \cdot (-c\nabla \tilde{u} - \alpha \tilde{u} + \gamma) . n ds + \int_{\Omega} \nabla \tilde{v} \cdot (-c\nabla \tilde{u} - \alpha \tilde{u} + \gamma) dA = \int_{\Omega} \tilde{v}\left(f - \beta \cdot \nabla \tilde{u} - a\tilde{u}\right) dA \tag{B-24}$$

Substitution the Neuman condition in equation (B-21) into equation (B-24) gives equation (B-25).

$$\int_{\partial\Omega} \tilde{v} \cdot (-q\tilde{u} + g - h^{T}\mu) ds + \int_{\Omega} \nabla \tilde{v} \cdot (-c\nabla \tilde{u} - \alpha \tilde{u} + \gamma) - \tilde{v}\left(f - \beta \cdot \nabla \tilde{u} - a\tilde{u}\right) dA = 0 \tag{B-25}$$

Equation (B-25) is the weak formulation of the finite element statement in equation (B-23) and is simply written as shown in equation (B-26).

$$\int_{\Omega} W_1 dA + \int_{\partial\Omega} W_2 ds - \int_{\partial\Omega} \tilde{v} h^{T} \mu ds = 0 \tag{B-26}$$

The test functions occur linearly in the integrands of the weak equation so it is enough to require that the weak equation holds the test functions are chosen as the basis functions (FEMLAB 2005) as shown in equation (B-27). When this is substituted in the weak formulation it gives one equation for each *i*.

$$\tilde{v} = \phi_i \quad \forall i = 1,.....N \qquad \text{(B-27)}$$

The Lagrange multiplier μ is also descretized and the matrix of equations assembled. It then solves for the solution vector and the Lagrange multiplier vector by using one of its solver algorithms[66].

B.5 Implementing Boundary conditions in 2DPM

It is necessary to document a special difficulty in using FEMLAB to implement the boundary conditions in the 2DPM model. The geometry object in this model is made of segments defined by the reserve-price curve. In a typical FEMLAB model, the number of boundary segments in the geometry is fixed as well as the index assigned to each boundary. In order to find the optimal exercise boundaries, the assumed boundaries are modified in every iteration until there is convergence. In this case, the locations of the exercise boundaries change the number of boundary segments in the whole model. The number of boundary segments in each subdomain, as well as, the index assigned to each boundary segment also changes. This required the ability to track the individual boundary segments and re-assign boundary indices taking into account any added or deleted segments and be able to apply the right boundary conditions.

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[66] See FEMLAB documentation for further details on the descretization of the Lagrange multiplier and the list of available solvers.

The above requirement was achieved by developing five functions *bndvectorOP(……), bndvectorCL(……), bndvectorDV(……), bndvector(……)* and *trackbnd(…..). Trackbnd(…..)* takes as parameters the *fem* structure and the name of the boundary segment. It searches through the *mesh object* to identify the boundary segments and their indices[67]. *BndvectorOP(……), bndvectorDV(……), bndvectorCL(……), bndvector(……)* generates the new boundary condition vector for the open, close, develop and invest options.

B.6 Implementing Boundary conditions MSS=0 and MRR=0 in 2DPR

The general form of the PDE formulation in equation (B-21) shows that FEMLAB can only implement first order conditions at the boundaries. In order to implement second order conditions, the second order condition was substituted in the governing equation since the governing equation must also satisfy the boundaries. In the 2DPM model it is easier to implement $M_{SS}=0$ since the substitution of $M_{SS}=0$ into the governing equations reduces it to a first order equation. It easy to implement this through the generalized *Neuman* condition by introducing some weak term contributions as explained in section 4.5.3. In the 2DPR model however, substituting $M_{SS}=0$ or $M_{RR}=0$ in the governing equation still leaves a second order term (M_{RR} or M_{SS}) in the governing equation as shown in equations (B-28) and (B-29) for an operating mine.

$$O_{SS} = 0 \equiv \left(-q + (\phi - \lambda_R)R\right)O_R + S(r - \delta)O_S + \frac{1}{2}\bar{\sigma}^2 R^2 O_{RR} - rO + A = 0 \qquad \text{(B-28)}$$

---

[67] I am very grateful to Dr. Erik Danielsson of COMSOL Multiphysics' Support at the Stockholm office for guiding me through the implementation of this code

$$O_{RR} = 0 \equiv \left(-q + (\phi - \lambda_R)R\right)O_R + S(r - \delta)O_S + \frac{1}{2}\sigma^2 S^2 O_{SS} - rO + A = 0 \qquad \text{(B-29)}$$

In order to overcome this, an imaginary boundary is introduced very close to the actual boundary. A boundary extrusion coupling variable is set equal to the second derivative term. For example very close to the maximum price boundary (in this case 0.1 units from the maximum) an extrusion coupling variable equal to M_{RR} is defined and made available at the maximum boundary. Equation (B-28) for example reduces to the form shown in equation (B-30) without the second order term, M_{RR}. K(R) is the value of M_{RR} close to the maximum price boundary. The assumption here is that if the functional form of *M* is smooth and twice differentiable, then the first and second derivatives are smooth and continuous with no abrupt changes in the derivatives for small changes in the state variables. It is then possible to implement the generalized *Neuman* condition by introducing the appropriate weak term contributions. The implementation is shown in equations (B-31) to (B-33).

$$O_{SS} = 0 \equiv \left(-q + (\phi - \lambda_R)R\right)O_R + S(r - \delta)O_S + \frac{1}{2}\bar{\sigma}^2 R^2 K(R) - rO + A = 0 \qquad \text{(B-30)}$$

$$\boldsymbol{g} = \frac{\left(-A - \frac{1}{2}K(R)\bar{\sigma}^2 R^2\right)\sigma^2 S^2}{2S(r - \delta)} \qquad \text{(B-31)}$$

$$\boldsymbol{q^*} = \frac{-r\sigma^2 S^2}{2S(r - \delta)} \qquad \text{(B-32)}$$

$$\text{weak} = \frac{(q-\phi R)O_R\sigma^2 S^2 O\_test}{2S(r-\delta)} \qquad \text{(B-33)}$$

B.7 Numerical Differentiation

The Taylor series expansion can be used to numerically derive expression for the finite divided difference approximation for derivatives. The derivatives could be approximated by either the forward, backward or central difference depending on the Taylor series expansion. Equation (B-34) shows the forward Taylor series expansion. The forward difference approximation of the first derivative is therefore given by equation (B-35).

$$f(x_i+h) = f(x_i) + hf'(x_i) + \frac{1}{2}h^2 f''(x_i) + \ldots. \qquad \text{(B-34)}$$

$$f'(x_i) = \frac{f(x_i+h) - f(x_i)}{h} - O(h) \qquad \text{(B-35)}$$

The forward difference approximation of the first derivative uses the *ith* and (*i+1*) data points to estimate the derivative. The backward and central difference approximations for the first derivative are also shown in equations (B-36) and (B-37). The approximation by the central difference is usually more accurate for the same step size *h.*

$$f'(x_i) = \frac{f(x_i) - f(x_i-h)}{h} - O(h) \qquad \text{(B-36)}$$

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h} - O(h^2) \qquad \text{(B-37)}$$

The second order derivative can also be derived by using the Taylor series expansions. This is derived by writing the Taylor's forward or backward expansion for $f(x_i+2h)$ in terms of $f(x_i)$ and combining with the expansion for $f(x_i+h)$. The forward second order derivative for example is derived by combining equations (B-34) and (B-38) to yield equation (B-39).

$$f(x_i+2h)=f(x_i)+(2h)f'(x_i)+\frac{1}{2}(2h)^2f''(x_i)+.... \qquad \text{(B-38)}$$

$$f''(x_i)=\frac{f(x_i+2h)-2f(x_i+h)+f(x_i)}{h^2}+O(h) \qquad \text{(B-39)}$$

The backward and central second derivatives are given by equations (B-40) and (B-41). The order of error of the central difference is higher as such it is again more accurate for the same interval *h*.

$$f''(x_i)=\frac{f(x_i)-2f(x_i-h)+f(x_i-2h)}{h^2}+O(h) \qquad \text{(B-40)}$$

$$f''(x_i)=\frac{f(x_i+h)-2f(x_i)+f(x_i-h)}{h^2}+O(h^2) \qquad \text{(B-41)}$$

MATLAB uses the central difference in approximating the first and second order derivatives by using the function ***differentiate (......)***.

B.8 Goodness of Fit Statistics[68]

B.8.1 Sum of Square Errors (SSE)

This statistic measures the total deviation of the response values from the prediction by the fitted model as defined in equation (B-42). It is also called the summed square of residuals and is usually labeled as SSE. A value close to zero is an indication of a better fit.

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \qquad \text{(B-42)}$$

B.8.2 R-Square

This statistic measures the degree to which the variation in the response data is explained by the fitted model. It is the square of the correlation coefficient between the response data and the response predicted by the fitted model. The value lies between *0* and *1.* A value close to zero indicates a poor fit. It is also called the coefficient of multiple determination. It is defined as the ratio of the sum of squares of regression (SSR) defined in equation (B-44) and the total sum of squares (SST) or sum of squares about the mean defined in equation (B-45).

$$\text{R-Square} = \frac{SSR}{SST} = 1 - \frac{SSE}{SST} \qquad \text{(B-43)}$$

$$SSR = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \qquad \text{(B-44)}$$

---

[68] Compiled from the Documentation in MATLAB (2004)

$$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2 \tag{B-45}$$

$$SST = SSR + SSE \tag{B-46}$$

B.8.3 Adjusted R-Square

This statistic adjusts the R-Square statistic to account for the degrees of freedom used in the fitted model. The residual degree of freedom (DOF) is defined as the number of response values *minus* the number of coefficients used to estimate the fitted model. It is calculated using equation (B-47) where *n* is the number of response values and *v* is the residual DOF.

$$\text{adjusted R-Square} = 1 - \frac{SSE(n-1)}{SST(v)} \tag{B-47}$$

The adjusted R-square statistic is generally the best indicator of the fit quality when and additional coefficients are added to a model.

B.8.4 Root Mean Square Error (RMSE)

This statistic is also known as the fit standard error and the standard error of the regression. It is the square root of the mean square error (MSE) defined in equation (B-49).

$$RMSE = \sqrt{MSE} \tag{B-48}$$

$$MSE = \frac{SSE}{v} \tag{B-49}$$